The Problem of Time in Quantum Cosmology: A Decoherent Histories View

Petros Wallden*

10th June 2010
NEB 14, Ioannina

*Department of Physics, University of Athens

- T. Christodoulakis & PW: in preparation
- PW: Int. J. Theor. Phys. **47**, 1512 (2008).
- J.J. Halliwell & PW: PRD **73,** 024011 (2006).

Contents

Problem of Time

Diffeomorphism invariance in GR Vs
Fixed parameter time in Newtonian Physics.

- Time in Quantum Theory:
  - Not Observable
  - Appears as a parameter
  - Physical clocks run backwards in abstract Newtonian Time
- Time in General Relativity:
  - How does 'change' appears?
  - Time is locally defined
  - How to make it compatible with QT that is based on Newtonian Time?

Technical Problem

- Constrained Systems:

  Less degrees of freedom

  e.g. in E.M.: $A_\mu \to A_\mu + \partial_\mu \phi$

  Physics cannot depend on choice of gauge

  Physical states are *equivalences classes*

- To quantize:

  (a) Constrain and THEN quantize

  (b) Quantize and THEN impose the constraint

- If the constraint in the Classical Theory is:

  $\phi(p, x) = 0$

  in the Quantum Theory becomes:

  $\widehat{\phi}|\psi\rangle = 0$

  and also require that observables $\hat{A}$ obey:

  $[\widehat{A}, \widehat{\phi}] = 0$

  We start with the *kinematical* Hilbert space (unconstrained) $\mathcal{H}_{kin}$.

  The physical states that obey the above condition form the *physical* Hilbert space $\mathcal{H}_{phys}$.

GR as Constrained System

The "gauge" in general relativity is the invariance of the theory under diffeomorphisms, Diff($\mathcal{M}$), which breaks into:

(a) Spatial "three"-dimensional diffeomorphisms

(b) *Hamiltonian* constraint: $\widehat{H}|\psi\rangle = 0$

$$i\hbar\frac{d\widehat{A}(t)}{dt} = [\widehat{H}, \widehat{A}(t)] = 0$$

for any $\hat{A}$ observable, due to the constraint:

Any observable $\hat{A}$, is independent of time!

• General feature of ANY theory that has vanishing Hamiltonian (in particular when the constraint is quadratic in *all* momenta, and it cannot be "deparametrized")

Timeless Theories

• Need to construct a Quantum Theory that time does not have any fundamental role.

• Time "emerges" as a coarse grained property of the relative field configurations.

> All physical questions can be translated to questions about the possible relative configurations of the universe and its material content.

(a) Evolving Constants

(b) Decoherent Histories

The Decoherent Histories Approach to QT

An alternative formulation of Quantum Theory designed to deal with closed systems. Among other things it aims to

(a) Assign probabilities to histories of closed system.

(b) Deal with time-extended questions.

(c) Put space and time in equal footing. Time is no longer in a preferred position, since we are dealing with whole histories of the system (rather than single time propositions).

Due it these facts, it suits well for dealing with the problem of time.

Decoherent Histories: Non-relativistic QM

Copenhagen probabilities for sequential measurements:

$P(\alpha_{t_1}$ at t_1 and α_{t_2} at $t_2 \cdots\ \alpha_{t_n}$ at t_n; $\rho(t_0))=$

$\mathrm{Tr}(\alpha_{t_n}(t_n)\cdots\alpha_{t_1}(t_1)\rho(t_0)\alpha_{t_1}(t_1)\cdots\alpha_{t_n}(t_n))$

This is NOT probability for closed system, fails to satisfy the "additivity of disjoint regions of the sample space", due to interference.

Under certain conditions this probability CAN be assigned to histories of closed systems.

Class operator: $C_{\underline{\alpha}} = \alpha_{t_n}(t_n)\cdots\alpha_{t_1}(t_1)$

Decohernece Functional (measures interference):

$\mathcal{D}(\underline{\alpha}, \underline{\alpha}') = \text{Tr}(C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\alpha}'})$

A set of histories $\{\underline{\alpha}_i\}$, that is *disjoint* and *exhaustive* is called *complete*.

Probabilities are assigned to a history α_i, provided it belongs to a complete set such that:

$\mathcal{D}(\underline{\alpha}_i, \underline{\alpha}_j) = 0$, for all $i \neq j$.

The probability is then $p(\underline{\alpha}_i) = \mathcal{D}(\underline{\alpha}_i, \underline{\alpha}_i)$.

• Typically, there exist more than one complete set that obeys the decoherence condition. There is some interpretational ambiguity.

• The above can be generalized (relativistic QT or quantum gravity). Construction of Class Operators that correspond to physical questions, and an inner product to define probabilities and the decoherence condition.

Decoherent Histories and the Problem of Time

Histories and Classical Timeless Questions:

> Does a (classical) trajectory cross a given region Δ of the configuration space? If it is the full trajectory, then this is indeed reparametrization invariant.

In the Quantum Case, we require also:

(i) Initial state has to obey: $\widehat{H}|\psi\rangle = 0$

(ii) Class operator: $[\widehat{C}_\alpha, H] = 0$

(iii) We have to use the *induced* (or Rieffel) inner product. (essentially an inner product defined on solutions of the constraint.)

Proposed Class Operator

What is the probability that the system crosses region Δ of configuration space, with no reference in time.

Need to find a Class Operator (CO) that commutes with the Hamiltonian and gives (semi-classically) sensible results.

Since the classical reparametrization invariant object is full trajectory we consider the unphysical parameter time running from $-\infty$ to $+\infty$.

CO Crossing $\Delta = 1$- CO Always in $\bar{\Delta}$

$$C_{\bar{\Delta}} = \prod_{t=-\infty}^{t=+\infty} \bar{P}(t)$$

$$[C_{\bar{\Delta}}, H] = 0$$

$$C_{\bar{\Delta}} = \lim_{t''\to\infty, t'\to -\infty} \exp(-iHt'') g_r(t'', t') \exp(iHt')$$

$$C_{\Delta} = \mathbf{1} - C_{\bar{\Delta}}$$

This expression resembles the arrival time problem in standard non-relativistic quantum mechanics (see J.J. Halliwell & E. Zafiris in PRD also PW in IJTP and recently Halliwell & Yearsley).

Note that for periodic Hamiltonians (bounded systems), the above analysis changes slightly (see details in J.J.Haliwell & PW).

Restricted Propagator

- Path Integral definition:

$$
\begin{aligned}
g_r(x, t \mid x_0, t_0) &= \int_{\bar{\Delta}} \mathcal{D}x \exp(iS[x(t)]) \\
&= \langle x | g_r(t, t_0) | x_0 \rangle \qquad (1)
\end{aligned}
$$

(the integral is over paths restricted to the region $\bar{\Delta}$)

- Operator definition (more general) $\delta t \to 0$, $n \to \infty$ keeping $\delta t \times n = (t - t_0)$:

$$
\begin{aligned}
g_r(t, t_0) &= \lim_{\delta t \to 0} \bar{P} e^{-iH(t_n - t_{n-1})} \bar{P} \dots \bar{P} e^{-iH(t_1 - t_0)} \bar{P} \\
&= \bar{P} \exp\left(-i(t - t_0) \bar{P} H \bar{P}\right) \bar{P} \qquad (2)
\end{aligned}
$$

- An important property:

$$
g_r^\dagger(t, t_0) g_r(t, t_0) = \bar{P} \qquad (3)
$$

General No-Crossing Probabilities and D.Condition

Using the last property we have (candidate) probability for not crossing:

$p_{\bar{\Delta}} = \langle\psi|C^{\dagger}_{\bar{\Delta}} C_{\bar{\Delta}}|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle$

$p_{\Delta} = 1 - p_{\bar{\Delta}} = \langle\psi|P|\psi\rangle$

Provided we have *decoherence*, while as decoherence condition we get:

$$\lim_{t\to\infty, t_0\to-\infty} e^{iE(t-t_0)}\langle\psi|g_r(t,t_0)|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle$$

which becomes the need to vanish on the boundary of the region, i.e.:

$\langle x|\psi\rangle = 0, \quad \forall x \in \partial\bar{\Delta}$

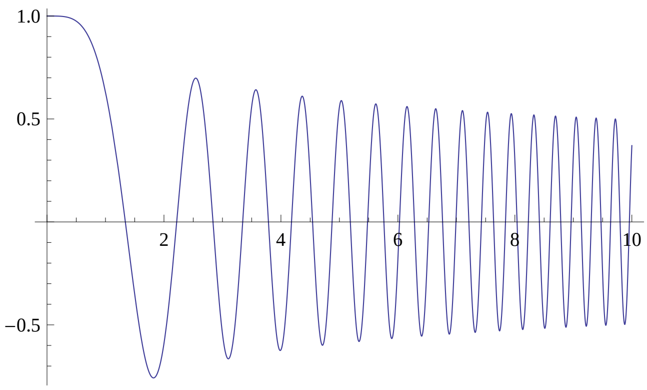
Example: FRW-QC

Homogeneous and isotropic universe with $k = 1$ (3-sphere). First (a) empty and then (b) with a scalar field with potential $V(\phi) = e^{2\phi}$.

First case:

$$\Psi''(\alpha)/4\alpha - \Psi'(\alpha)/8\alpha^2 + \alpha\Psi(\alpha) = 0$$

The solutions are Bessel functions we take one of them, $\Psi(\alpha) \propto \alpha^{3/4} J_{-\frac{2}{3}}(\alpha^2)$ with graph



And can ask the probability that it never crosses the region $\alpha > 6$. It coincides with a zero of the Bessel function and thus decoheres. The probability turns out to be $p_c \simeq 0.27$

Second case (with scalar field). Wheeler De-Witt equation

$$(2\alpha - \alpha^3 e^{2\phi})\Psi(\alpha,\phi) - \frac{1}{2\alpha}\frac{\partial^2\Psi(\alpha,\phi)}{\partial\alpha^2}$$
$$-\frac{1}{2\alpha^2}\frac{\partial\Psi(\alpha,\phi)}{\partial\alpha} + \frac{1}{2\alpha^3}\frac{\partial^2\Psi(\alpha,\phi)}{\partial\phi^2} = 0$$

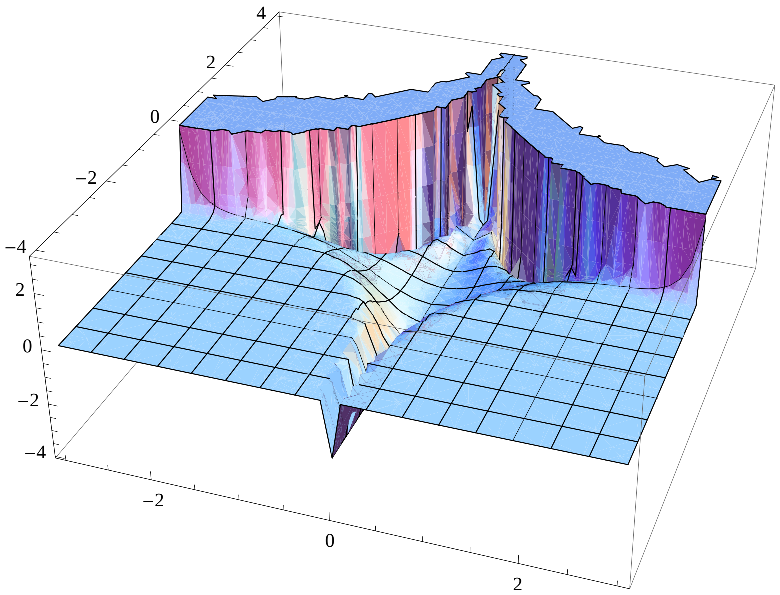
The general solution is

$$\Psi(\alpha,\phi) = C_2 \exp\frac{\alpha^2 e^{-2\phi}(-4C_1^2 - 4e^{4\phi} + 6e^{6\phi}\alpha^2)}{8C_1}$$

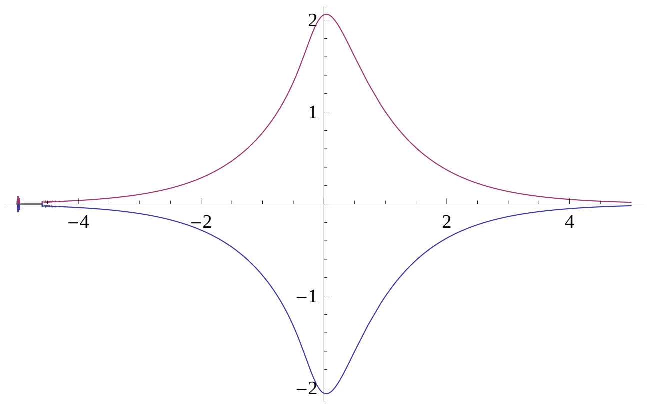
Consider one solution (note that it is NOT normalizable in the normal inner product, and we need to use an inner product on solutions)

$$\begin{aligned}\Psi(\alpha,\phi) \;=\; & \exp\frac{\alpha^2 e^{-2\phi}(-4^2 - 4e^{4\phi} + 6e^{6\phi}\alpha^2)}{8} - \\ & \exp\frac{\alpha^2 e^{-2\phi}(-4\cdot 1.5^2 - 4e^{4\phi} + 6e^{6\phi}\alpha^2)}{12}\end{aligned}$$

Which looks like



We can ask which is the probability that the universe never crosses the region defined by:



Since our solution vanish at the above boundary, we have decoherence and we can assign the crossing probability (note that the integral $|\Psi(\alpha, \phi)|^2$ at the region is no-zero, however in the induced inner product it results to zero probability) $p_c = 0$.

Other solutions (involving superpositions) or systems with more degrees of freedom (e.g. Bianchi Cosmologies, different matter content), result to non-trivial questions.

Compariton with relational observables: Choose questions like "value(s) of α when $\phi = 15$ or when $\phi > 15$". This corresponds to an observable that projects at the range of ϕ in question. The resulting operator does NOT commute with Hamiltonian. Fails only on the boundary of the region.

If one restricted attention to a single solution (not formally allowed at this approach) that vanishes at this boundary, he would recover exactly our result.

Summary & Conclusions

• We examined the DH analysis of timeless QT. We got Class Operators that respect the Hamiltonian constraint.

• They consisted of a general enough set of physical questions of the type:

"Which is the prob that it crosses a region in configuration space with no reference in time"

• We have got an easy but restrictive decoherence condition "The initial state has to vanish on the boundary of the region considered"

• The probabilities for those histories are easily calculated.

• We considered as an example the case of FRW Quantum Cosmology with scalar field. Given a solution of the Wheeler-DeWitt equation we found questions that can be answered and compared with the evolving constants.