Black holes from generalized gauge field theories

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Outline

Some developments in gravitating field configurations

- Hoffmann and Infeld (1935-37) found solutions to the Einstein equations coupled to non-linear electrodynamic models (e.g. Born-Infeld: $L = \beta^2(1 - (1 - \frac{\vec{E}^2}{\beta^2})^{1/2}) \rightarrow$ Energy finite!)
- Eighties: Renewed interest in the topic, partially motivated by some low-energy results of string theory.
  - Black hole solutions: Several NEDs coupled to GR (*e.g. Plebanski 84, Demianski 86, Oliveira 94, Gibbons 95, Rasheed 97...*)
- Other developments include:
  -Coupling to gravity can remove the restrictions of some non-existence theorems of solitons in flat space (*Bartnik and MacKinnon 88)*
  - Black hole configurations in (Anti-)de Sitter spaces (*e.g. Dey'04, Cai'04*): motivated by the AdS/CFT conjecture. Topological black holes.
  - Higher-order gravity theories with NEDs (*Aiello 04*): suggested by string theory...
  - Black holes in non-abelian generalized gauge field theories (*Volkov 99, Dyadichev 00, Wirschins 01, etc)*

Characterization of the models in flat space

- Non-linear electrodynamics (NED): An arbitrary function

$$L = \varphi(X, Y)$$

of the two standard field invariants
$X = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{B}^2$, $Y = -\frac{1}{2}F_{\mu\nu}F^{*\mu\nu} = 2\vec{E} \cdot \vec{B}$

- $\varphi(X, Y)$ restricted by some conditions (*"admissibility"*)
  1. φ must be a continuous, derivable and single-valued function on its domain of definition of the $X - Y$ plane
  2. Parity invariance $\varphi(X, Y) = \varphi(X, -Y)$
  3. Positive definite character of energy for *any* field configuration $\rho \geq \left(\sqrt{X^2 + Y^2} + X\right)\frac{\partial\varphi}{\partial X} + Y\frac{\partial\varphi}{\partial Y} - \varphi(X, Y) \geq 0$

$\rightarrow [E(r) \neq 0, B = 0]$ (ESS fields) determined through a first-integral

$$r^2\varphi_X E(r) = q$$

- Convergence of the energy ((3+1)-dim) of the ESS field

$$\varepsilon(q) = \int_0^\infty r^2 T_0^0(r,q)dr = q^{3/2}\varepsilon(q=1)$$

depends on the behaviour of $r^2 T_0^0(r,q) \sim E(r)$ at $r \to \infty$ and around $r \sim 0$. $\rightarrow$ Classification of NED models into families of finite-energy ESS fields and divergent-energy ESS fields.

I) Finite-energy ESS solutions

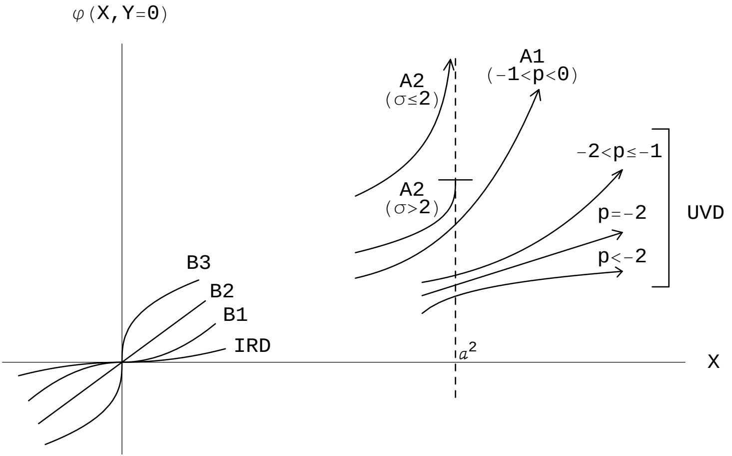
- $r \to \infty$: $E(r) \sim r^p, p < -1$. Three subcases:
  1. $-2 < p < -1$: Slower than coulombian (case B1)
  2. $p = -2$: Usual Coulombian behaviour (case B2)
  3. $p < -2$: Faster than coulombian (case B3)
- $r \sim 0$: $E(r) \sim r^p, -1 < p \leq 0$. Two subcases:
  1. A1: $E(r) \sim 1/r^p, -1 < p < 0$
  2. A2: $E(r) \sim a - br^\sigma (p = 0)$

II) Divergent-energy ESS solutions

Two classes:

- **UVD** case: The field diverges around $r \sim 0$ as $E(r) \sim \beta r^p, p < -1$ but converges at $r \to \infty$ (B-field). Example: Maxwell theory $\varphi(X) = X$ ($E(r) \sim \beta r^{-2}$)
- **IRD** case: The field vanishes at $r \to \infty$ as $E(r) \sim \beta r^p$, $-1 \leq p < 0 \to \varepsilon$ diverges there but converges around $r \sim 0$ (A-field).
  Examples: A1-IRD: $\varphi(X) = \beta X^\gamma, \gamma > 3/2$
  A2-IRD: $E(r) = \frac{1}{(r^2/q+\mu^2)^{1/2}}$

The behaviours of the admissible lagrangian densities $\varphi(X) = \varphi(X, Y = 0)$ are summarized in this plot ($E(r) \sim r^p$):



Einstein-NED spherically symmetric solutions

- Action: $S = S_G + S_{NED} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \varphi(X,Y)\right]$
- Source symmetry T_0^0 leads to a SS line element

$$ds^2 = \lambda(r)dt^2 - \lambda^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\vartheta^2)$$

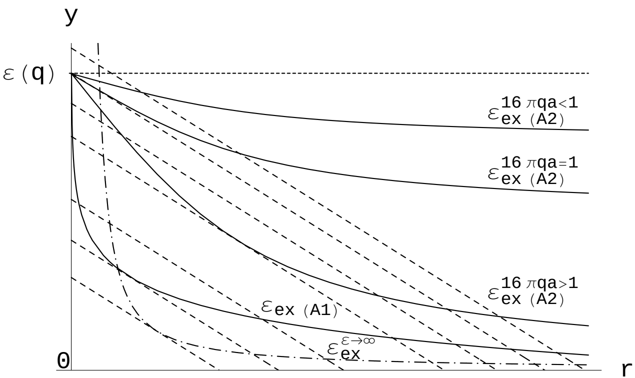
- The first-integral remains unmodified in the gravitational context. Also $X = E(r)^2$ does not depend on $\lambda(r)$.
- Integration of the metric leads to

$$\lambda(r) = 1 - \frac{2M}{r} + \frac{2\varepsilon_{ex}(r)}{r}$$

($\varepsilon_{ex}(r,q) = 4\pi\int_r^\infty R^2 T_0^0(R,q)dR$: exterior integral of energy, a **monotonically decreasing and concave function of r**)

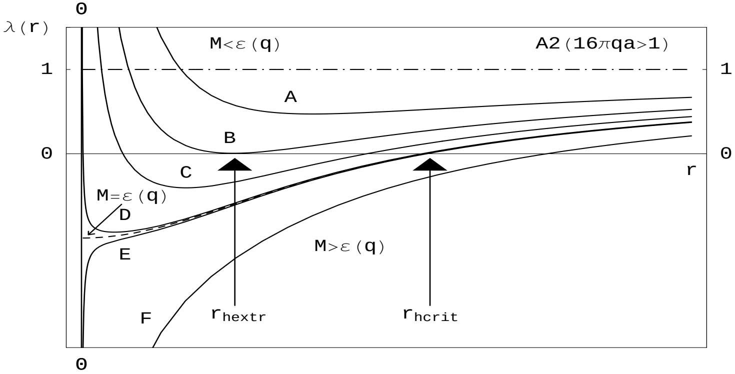
- Horizons: $\lambda(r_h) = 0 \rightarrow M - \frac{r_h}{2} = \varepsilon_{ex}(r,q)$

- Horizons given by the cut points of the curves $y = \varepsilon_{ex}(r, q)$ with the beam of straight lines $y = M - r_h/2$



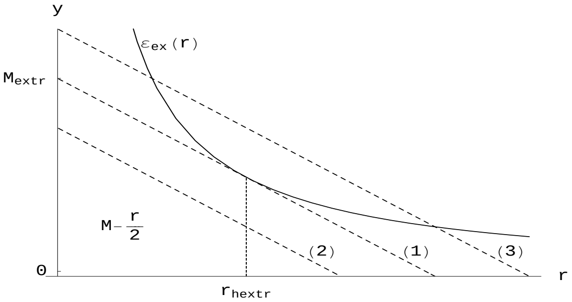
- Extreme black holes: radius $8\pi r_{hextr}^2 T_0^0(r_{hextr}, q) = 1$, mass: $M_{hextr}(q) = \frac{r_{hextr}(q)}{3} + \frac{16\pi q}{3} A_0(r_{hextr}, q)$
  $\rightarrow$ Available for cases A1, A2 ($16\pi qa \geq 1$)

- The behaviour of the metric around $r \sim 0$ depends on the sign of $M - \varepsilon(q)$
  Example: In the case A2 with $16\pi qa > 1$:



- "Finite-metrics" (around $r \sim 0$) only exist for fields A2: $\lambda(0) \to 1 - 16\pi qa$

Divergent-energy family (I): UVD + B-Field



1. $M = M_{extr}(q)$: Extreme black hole
2. $M < M_{extr}(q)$: No horizons (naked singularity)
3. $M > M_{extr}(q)$: Two-horizon BH (event and Cauchy)

Divergent-energy family (II): IRD + A-Field

- Since the energy diverges at infinity, $\varepsilon_{ex}(r,q)$ is not well defined: the metric is integrated as (C: integration constant)

$$\lambda(r) = 1 + \frac{C}{r} - \frac{2\varepsilon_{in}(r,q)}{r}$$

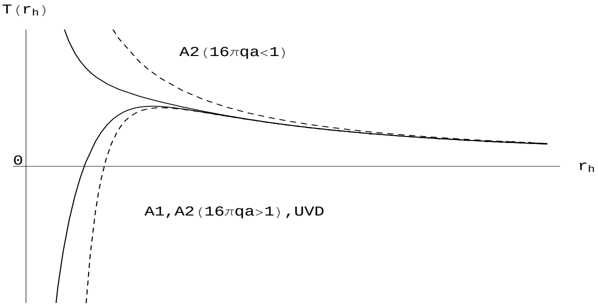
where

$$\varepsilon_{in}(q,r) = 4\pi \int_0^r R^2 T_0^0(R,q) dR$$

monotonically increasing and convex

- Similar classification procedure as in the finite-energy cases, depending on the sign of C.
  $\rightarrow$ They tend to 1 at $r \rightarrow \infty$ slower than the Schwarzschild solution

Temperature: $T = \frac{1}{4\pi}\frac{dg(r)}{dr}\Big|_{r=r_h} = \frac{1}{4\pi r_h}(1 - r_h^2 T_0^0(r_h, q))$



- T always positive for A2 ($16\pi qa > 1$): similar to the Schwarzschild solution.
- A1, A2 ($16\pi qa > 1$), UVD: RN-like behaviour
- Critical case ($16\pi qa = 1$): T at $r_h \to 0$ can diverge, vanish or take a (positive) finite value.

Extension to non-abelian fields

- Taking the two standard first-order field invariants $X = -\frac{1}{2} F^a_{\mu\nu} F^{\mu\nu a}$, $Y = -\frac{1}{2} F^a_{\mu\nu} F^{*\mu\nu}_a$, $a = 1 \cdots N$.
- Configurations $A^a_0 \neq 0, A^a_i = 0, \forall$ a lead to N first-integrals $(X = \sum_{a=1}^n (E^a)^2)$

$$r^a \varphi_X E^a = q^a$$

- Similar procedure of metric integration as for the abelian case, and the solution is the same under the replacement:
  $q \rightarrow Q = \sqrt{\sum_{a=1}^N (q^a)^2}$ "mean-square" charge
  $\rightarrow \vec{E}^a = \frac{q^a}{Q} \vec{E}(r)$
  $\lambda(r) = 1 - \frac{2M}{r} + \frac{2\varepsilon_{ex}(r,Q)}{r}; \varepsilon_{ex}(r, Q) = 4\pi \int_r^\infty R^2 T_0^0(R, Q) dR$

NED in Gauss-Bonnet theory

- Gauss-Bonnet: (units $16\pi G = 1$)

$$S = \int d^{n+1}x\sqrt{-g}\Big[(R-2\Lambda) + \\ + \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\Big] + S_{NED}$$

- Einstein equations lead to a relation $g_\alpha(r) - g_0(r) = \frac{l_\alpha^2}{2r^2}(1-g_\alpha(r))^2$, $l_\alpha^2 \propto (n-2)(n-3)\alpha$, where $g_0(r) = 1 - \frac{m}{r^{n-2}} + \frac{\varepsilon_{ex}(r,q)}{r^{n-2}}$: solution with $\alpha = 0$
- Generalization to $(n+1)$-dim: $\varepsilon_{ex}(r,q) \propto \frac{2}{n-1}\int_r^\infty R^{n-1}T_0^0(R,q)dR$

- Solution of the Einstein equations:

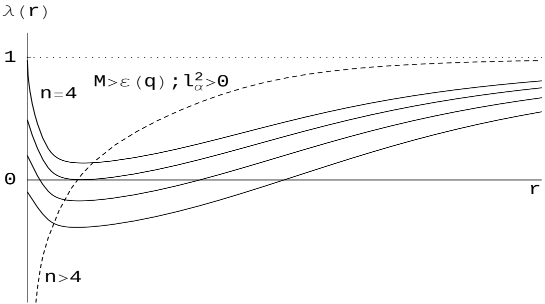
$$g_\alpha(r) = 1 + \frac{r^2}{l_\alpha^2}\left(1 - \left[1 + \frac{4l_\alpha^2}{r^n}\left(M - \varepsilon_{ex}(r,q)\right) - \frac{2l_\alpha^2}{l_\Lambda^2}\right]^{1/2}\right)$$

- There is still a first-integral $r^{n-1}\varphi_X E(r) = q$
- Energy finiteness conditions ($\varepsilon(q) \propto \int_0^\infty r^{n-1} T_0^0(r,q)dr$) in $n > 3$ easily obtained. Main conclusions:
  - B1, B2, B3 and A2 classes remain unmodified
  - A1 class: $E(r) \sim r^p$ convergence of $\varepsilon(q)$ depends on n
- Many new possibilities, depending on $\Lambda \gtreqless 0$, M, $\varepsilon(q)$, α, n... e.g. three-horizon black holes, branch singularities...

Conclusions and open problems

- These methods allow the analysis of general NED models without explicitly fixing the lagrangian function, and lead to:
  1. For NEDs with energy-divergent (in flat-space) ESS solutions, structure of gravitating solutions is similar as the RN case, or approach asymptotic flatness slower than Schwarzschild.
  2. For NEDs with finite-energy (in flat-space) ESS solutions, qualitatively different features appear, e.g. single horizon (non-extreme) black holes and "black points" ($r_h \to 0$).
  3. In higher-order gravity theories many other solutions arise: classified also depending on $M - \varepsilon(q) \gtreqless 0$.
- Some open problems:
  1. Thermodynamic analysis of these solutions, phase transitions? (work in progress)
  2. Stability: Existence of some general criteria? Flat space: $\frac{\partial \varphi}{\partial X} - 2X \frac{\partial^2 \varphi}{\partial Y^2} \geq 0 \to$ Generalizable to GR? (work in progress)
  3. Regular solutions in $\varphi(R, X, Y)$ theories?

GB-NED. Example: Case with $\alpha > 0$ and mass M larger than the ESS field energy $\varepsilon(q)$ appropriately generalized to n dimensions.



Behaviour depends on the dimension. $n > 4$: "Schwarzschild-like" behaviour; $n = 4$ special case: the metric takes a finite value at the origin, leading to naked singularities, extreme black holes, two-horizons black holes or single-horizon black holes