FRW models in the conformal frame of $f(R)$ gravity

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Outline

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$$p = (\gamma - 1)\rho, \quad 0 < \gamma \leq 2.$$

3. The potential function $V(\phi)$ of the scalar field is bounded from below, but otherwise, arbitrary.

Motivation

- Higher order gravity theories (HOG) derived from Lagrangians of the form

$$L = f(R)\sqrt{-g} + 2L_{\mathrm{m}}(\Psi),$$

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f is an arbitrary smooth function and $L_{\mathrm{m}}(\Psi)$ is the matter Lagrangian.
- Under the conformal transformation

$$\widetilde{g}_{\mu\nu} = f'(R)\, g_{\mu\nu},$$

the field equations reduce to the Einstein field equations with a scalar field as an additional matter source

$$\widetilde{G}_{\mu\nu} = T_{\mu\nu}(\widetilde{g}, \phi) + \widetilde{T}_{\mu\nu}(\widetilde{g}, \Psi),$$

where

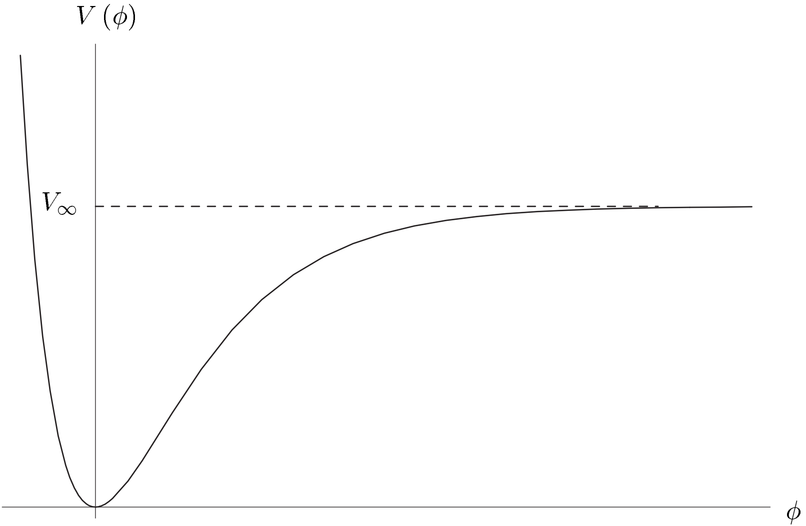
$$T_{\mu\nu}(\widetilde{g}, \phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\widetilde{g}_{\mu\nu}\left[(\partial\phi)^2 - 2V(\phi)\right],$$

and

$$\phi = \sqrt{\frac{3}{2}}\ln f'(R), \quad V(R(\phi)) = \frac{1}{2(f')^2}\left(Rf' - f\right).$$

Example: Potential arising in the conformal frame of the $R + \alpha R^2$ theory.

$$V(\phi) = \frac{1}{8\alpha}\left(1 - e^{-\sqrt{2/3}\phi}\right)^2$$



Remark

Bianchi identities imply

$$\widetilde{\nabla}^{\mu}\widetilde{T}_{\mu\nu}\left(\widetilde{g},\Psi\right)\neq 0,\quad \widetilde{\nabla}^{\mu}T_{\mu\nu}\left(\widetilde{g},\phi\right)\neq 0,$$

and therefore there is an energy exchange between the scalar field and ordinary matter.

Homogeneous and isotropic spacetimes

- The field equations reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

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Set $\frac{4-3\gamma}{\sqrt{6}} = \alpha$

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- Energy of the scalar field

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \Rightarrow \quad \dot{\epsilon} = -3H\dot{\phi}^2 + \alpha\rho\dot{\phi}.$$

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- The function

$$W(\phi, \dot{\phi}, \rho, H) = H^2 - \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho\right),$$

obeys

$$\dot{W} = -2HW.$$

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- The equilibria of the system have the form

$$(\phi = \phi_*, y = 0, \rho = 0, H = \sqrt{V(\phi_*)/3}),$$

where $V'(\phi_*) = 0$.

Asymptotically stable equilibrium

Proposition

Let ϕ_* a strict local minimum for $V(\phi)$, possibly nondegenerate, with nonnegative critical value. Then, $\mathbf{p}_* = (\phi_*, \dot{\phi}_* = 0, \rho_* = 0, H_* = \sqrt{\frac{V(\phi_*)}{3}})$ is an asymptotically stable equilibrium point for expanding cosmologies in the open spatial topologies $k = 0$ and $k = -1$.

Sketch of the proof.

The proof consists in constructing a compact set Ω in $\mathbb{R}^4$ and showing that it is positively invariant. Applying LaSalle's invariance theorem to the functions W and $(\rho + \epsilon)$ in Ω, it is shown that every trajectory in Ω is such that $HW \to 0$ and $H(\dot{\phi}^2 + \gamma\rho) \to 0$ as $t \to +\infty$, which means $\dot{\phi} \to 0$, $\rho \to 0$, and $H^2 - \frac{1}{3}V(\phi) \to 0$. Since H is monotone and admits a limit, $V(\phi)$ also admits a limit, $V(\phi_*)$, thus the solution approaches asymptotically the equilibrium point $\mathbf{p}_*$. □

Details in [R. Giambò and JM, Class. Quant. Grav. 2010]

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Similar results were proved in [JM, Class. Quant. Grav. 2003] for separately conserved scalar field and perfect fluid.

Energy exchange

We assume that ϕ_* is a nondegenerate minimum of $V(\phi)$ with null critical value, (for the sake of simplicity we will suppose $\phi_* = 0$)

$$V(\phi) = \frac{1}{2}\omega^2\phi^2 + \mathcal{O}(\phi^3), \qquad \omega > 0.$$

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and we are asking whether there is a time t_1 such that

$$\rho(t) > \epsilon\left(t\right), \qquad \forall t > t_1.$$

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If $V\,(\phi_*) > 0$, the transition does not happen. In that case, the energy of the scalar field tends to this value, $V\,(\phi_*) > 0$, whereas the energy of the fluid tends to zero.

Integrating the ρ equation

$$\rho(t) = ce^{-\alpha\phi(t)}a(t)^{-3\gamma} \quad \Rightarrow \quad \rho(t) \simeq ca(t)^{-3\gamma} \quad \text{as} \quad t \to \infty$$

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The equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + \omega^2\phi + \mathcal{O}(\phi^2) = \alpha\rho,$$

can be solved by the Kryloff-Bogoliuboff approximation.

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$$\ddot{\phi} + \eta f\left(\phi, \dot{\phi}\right) + \omega^2\phi = 0, \quad 0 < \eta \ll 1.$$

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For $\eta = 0$, the solution is that of the simple harmonic oscillator,

$$\begin{aligned} \phi(t) &= A \sin(\omega t + \chi) \\ \text{and} \quad \dot{\phi}(t) &= \omega A \cos(\omega t + \chi). \end{aligned}$$

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We are looking for a solution which resembles to the form of the simple harmonic oscillator

$$\begin{aligned} \phi(t) &= A(t) \sin(\omega t + \chi(t)) \\ \text{and} \quad \dot{\phi}(t) &= \omega A(t) \cos(\omega t + \chi(t)). \end{aligned}$$

Setting $\theta(t) = \omega t + \chi(t)$ and substituting into the DE

$$\frac{dA}{dt} = -\frac{\eta}{\omega} f\left(A\sin\theta, \omega A\cos\theta\right)\cos\theta,$$
$$\frac{d\chi}{dt} = \frac{\eta}{\omega A} f\left(A\sin\theta, \omega A\cos\theta\right)\sin\theta.$$

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Replace the RH sides by their *average values* over a range of 2π of θ, i.e. the amplitude $A(t)$ is *regarded as a constant* in taking the average,

$$\frac{dA}{dt} = -\frac{\eta}{2\pi\omega}\int_0^{2\pi} f(A\sin\theta, \omega A\cos\theta)\cos\theta d\theta,$$
$$\frac{d\chi}{dt} = \frac{\eta}{2\pi\omega A}\int_0^{2\pi} f(A\sin\theta, \omega A\cos\theta)\sin\theta d\theta.$$

Apply the KB approximation to our equation

$$\ddot{\phi} + 3H\dot{\phi} + \omega^2\phi + \mathcal{O}(\phi^2) = \alpha\rho.$$

we find for the amplitude of ϕ

$$\frac{dA}{dt} = -\frac{3}{2}HA + c\frac{\alpha^2 A}{2\omega}a^{-3\gamma} + \mathcal{O}(A^3),$$

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Integrating

$$A = Ca^{-3/2}g\left(t\right),$$

where $\lim_{t\to\infty} g\left(t\right) = 1$, therefore

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Since the amplitudes of ϕ and $\dot{\phi}$ have the same time dependence, $\simeq a^{-3/2}$, we conclude that

$$\epsilon \simeq \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\phi^2 \simeq a^{-3}.$$

Comparing $\epsilon \simeq a^{-3}$ with $\rho \simeq a^{-3\gamma}$, we arrive at the following conclusion.

If $\gamma < 1$ the energy density ρ eventually dominates over the energy density of the scalar field ϵ and this universe follows the classical Friedmannian evolution. For $\gamma > 1$, ϵ eventually dominates over ρ.

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A rigorous proof can be found in [R. Giambò and JM, Class. Quant. Grav. 2010].

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- In case the minimum is positive, say $V(\phi_*) > 0$, the energy density, ϵ, of the scalar field eventually rules over the energy density of the fluid, ρ, and the asymptotic state has an effective cosmological constant $V(\phi_*)$.
- In case the minimum is zero and nondegenerate, then ρ eventually dominates over ϵ if $\gamma < 1$ and ϵ dominates over ρ if $\gamma > 1$.
- These results could be interesting in investigations of cosmological scenarios in which the energy density of the scalar field mimics the background energy density. For viable dark energy models, it is necessary that the energy density of the scalar field remains insignificant during most of the history of the universe and emerges only at late times to account for the current acceleration of the universe.

- The above results can be rigorously proved only assuming that critical points are finite, and that $V(\phi)$ is non-negative as $\phi \to \pm\infty$. It must be remarked that the latter assumption does not enter in the study of the late time behavior around a critical point ϕ_*, because for that situation only the behavior of the potential near ϕ_* is important and no growth at infinity assumptions on V are actually needed.

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- The cases studied of course do not cover all possible situations. In particular, one can take into account degenerate minima for the potential.
- Another important question that should be further investigated is the case of closed cosmologies. We believe that a closed model cannot avoid recollapse, unless the minimum of the potential is strictly positive. In that case, the asymptotic state must be de Sitter space.