Review of the work on the representation theory of BMS group and its variants

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1 OUTLINE

- **BMS Group : History and Generalizations**

- **Results on the representation theory of the BMS Group in the Hilbert and nuclear topologies**

- **Research Programme**

- **Link with (ALE) instantons ?**

- **Results on the representation theory of B(2,2) and H(2,2)**

- **Open questions**

2 History and Structure

2.1 How did all start ?

- In 1962 Bondi and coworkers , Bondi H. Van der Burg , Metzner Proc. R. Soc. Lond. A. 269, 21-52 & Sachs Proc. R. Soc. Lond. A. 270, 103-126 , posed the question

- Does gravitational radiation carry away mass from the source?

- Answer : **Yes**
- The Model :

$$
\begin{aligned}
ds^2 &= (\frac{V}{r}e^{2\beta} - r^2U^2e^{2\gamma})du^2 + 2e^{2\beta}dudr \\
&+ 2r^2Ue^{2\gamma}dud\theta \\
&- r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2)
\end{aligned}
$$

- The coordinates are adapted to the geometry
  1. u=t-r represents the retarded time : is constant on outgoing null hypersurfaces
  2. θ and ϕ are the usual angular coordinates
  3. r is a radial coordinate which runs along the outgoing null geodesics
  4. β, γ, U and V are functions of $u, r,$ and θ
- future null infinity : $u =$ const. and $r \rightarrow +\infty$

- Sommerfield's outgoing radiation wave condition : γ (and also $\beta, U,$ and V,) should admit at future null infinity an expansion of the form

$$\gamma = \frac{f(t-r)}{r} + \frac{g(t-r)}{r^2} + ...$$

- Main result : Gravitational radiation carries away the mass of the source

- This is to be juxtaposed with electrodynamics : Electromagnetic radiation does not carry away the charge of the source

2.2 Unexpected spin-off : The BMS Group

- Question : Find " asymptotic isometries " at future null infinity, i.e., solve

$$\mathcal{L}_\xi g^{ij} = 0$$

when $u =$ const. and $r \to +\infty$

- The space-time is asymptotically flat, the curvature dies-off as one recedes from the source . Expectation : The "asymptotic isometry" group is the Poincare group

- Surprise : It is not the Poincare group but it is a much larger group , infinite dimensional, the BMS group

2.2.1 Description of the BMS Group

- The solutions of
$$\mathcal{L}_\xi g^{ij} = 0$$
at future null infinity are :
$$\begin{aligned}\bar{u} &= \frac{u + \alpha(\theta, \phi)}{K(\theta, \phi)} \\ \bar{\theta} &= H(\theta, \phi) \\ \bar{\phi} &= I(\theta, \phi)\end{aligned}$$
These mappings are called BMS transformations . They form a group , the BMS Group

- $a(\theta, \phi)$ is an arbitrary C^2 function on the S^2, and,
$\bar{K}^2(H, I)(dH^2 + \sin^2 H dI^2) = d\theta^2 + \sin^2\theta d\phi^2$
Therefore, H and I , represent a conformal mapping of the S^2 onto itself

2.2.2 Structure of the BMS Group

- The group of orientation preserving conformal self-transformations of S^2 is isomorphic with the connected component of the identity $L_+^\uparrow$ of the Poincare Group

- $B = \mathcal{A} \circledS_T L_+^\uparrow$
  1. $\mathcal{A}$ is the abelian group of real valued functions on the unit sphere S^2 under pointwise addition
  2. The semi-direct product B is taken with respect to the homomorphism $T : L_+^\uparrow \to Aut\mathcal{A}$ given by
  $$(T(\Lambda)\alpha)(p) = K_{\Lambda^{-1}}(p)\alpha(\Lambda^{-1}p),$$
  where, $\Lambda \in L_+^\uparrow$, $p \equiv (\theta, \phi) \in S^2$,and, $\alpha(p) \equiv \alpha(\theta, \phi) \in \mathcal{A}$

3. The conformal action : $p \longrightarrow \Lambda p$

In Minkowski space (t,x,y,z) take the future null cone $\mathcal{N}^+$. Associate with each line generator of $\mathcal{N}^+$ the point of intersection of this line with spacelike hyperplane t=1. This set of points is S^2. Since $L_+^\uparrow$ takes line generators into line generators , this gives an action of $L_+^\uparrow$ on S^2, the conformal action, $p \longrightarrow \Lambda p$. Let $l^\mu = (1, m^1, m^2, m^3)$ be a null vector. Then (m^1, m^2, m^3) is a point $p \in S^2$ and the conformal action $p \longrightarrow \Lambda p$ is given by

$$m^i \longrightarrow (\Lambda^0_\mu l^\mu)^{-1}(\Lambda^i_\mu l^\mu)$$

The conformal factor associated with the action is

$$K_\Lambda(p) = \Lambda^0_\mu l^\mu$$

•

- A complete orthonormal basis for $\mathcal{A}$ may be constructed from the normalised spherical harmonics $P_{lm}(\theta, \phi)$

$$\alpha(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{lm} P_{lm}(\theta, \phi)$$

where, $\bar{a}_{lm} = (-1)^m a_{l,-m}$

- $$\mathcal{A} = \mathcal{V} \bigoplus \Sigma$$

  where,

  $$\begin{aligned} \mathcal{V} : v(\theta, \phi) &= \sum_{l=0}^{1} \sum_{m=-l}^{m=l} a_{lm} P_{lm}(\theta, \phi) \\ &= a^0 + a^1 \sin\theta \cos\phi + a^2 \sin\theta \sin\phi \\ &+ a^3 \cos\theta \end{aligned}$$

  is the 'translation' subspace , and,

  $$\Sigma : \sigma(\theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{m=l} a_{lm} P_{lm}(\theta, \phi)$$

  is the 'supertranslation' space

- $\mathcal{V}$ is invariant under the action T of $L_{+}^{\uparrow}$ on $\mathcal{A}$, but Σ is not. Thus $\mathcal{A} = \mathcal{V} \bigoplus \Sigma$ is not preserved by the T action

- Interesting facts about B
  1. B is remarkably similar to P
     (a) $P/V = L_+^\uparrow$
     (b) $B/\mathcal{A} = L_+^\uparrow$
  2. $P \lhd B$
  3. $P \ntrianglelefteq B$
  4. B has two Abelian normal subgroups
     (a) One is four-dimensional : $V \trianglelefteq B$
     (b) One is infinite-dimensional : $\mathcal{A} \trianglelefteq B$
  5. $V \trianglelefteq \mathcal{A}$
  6. $PST = \mathcal{A}/V$
     $PST \ntrianglelefteq B$

- I $= B$ / V
  1. I is an infinite-dimensional group
  2. $PST \trianglelefteq I$
  3. $I/PST = L_+^\uparrow$

- “Internal” symmetries versus “space-time” symmetries : Komar (Phys. Rev. Lett. **15**,76(1965)) argued that the “spin-index” which parameterizes the IRS of I is the “isospin” of elementary particles . This proved to be false by McCarthy P.J. (Proc. R.Soc.Lond.A, **333**, 317, (1973))

2.3 B as a transformation group $B \times \Im^+ \to \Im^+$

- **Main result** : Penrose (Phys.Rev.Lett.**10**, 66, 1963) interpreted B as a transformation group $B \times \Im^+ \to \Im^+$ of the "future null infinity" of the space-times involved. Furthermore, he gave a geometric structure to $\Im^+$, the "strong conformal geometry" , such that this classical action $B \times \Im^+ \to \Im^+$ is the group of automorphisms of the geometry

- Motivation for dealing with null infinity , rather than spacelike infinity, is to obtain a description of the given system in terms of what is perceived by very distant observers ; such observers receive information about the change in state of the system along null geodesics rather than spacelike ones

- In this context it is very useful to attach to the original physical space-time manifold the set $\Im^+$ of future end points of null rays escaping form the system

- Penrose accomplished this by taking the conformal structure of space-time as fundamental . By doing so, he succeeded, at the same time, in defining isolated systems in General Relativity in a beautiful geometrical way

- The key observation is that “infinity” is far away with respect to the physical space-time metric. This means that one needs infinitely many “meter sticks” in succession to “get to infinity”

- But what if we replaced these meter sticks by ones that grow in length the farther out we go? Then it might be possible that only a finite number of them suffices to cover an infinite range , provided the growth rate is just right

- Make this idea more precise : Instead of using the physical space-time metric $\bar{g}$ to measure distance and time , we use a different metric $g = \Omega^2 \bar{g}$ which is "scaled down" with a scale factor Ω

- If Ω can be arranged to approach zero at an appropriate rate then this might result in "bringing infinity in to a finite region" with respect to the unphysical metric g

- We can imagine attaching points to the space time that are finite with respect to g but which are at infinity with respect to $\bar{g}$

- In this way we can construct a boundary $\Im$ consisting of all the end points of the succession of finitely many rescaled meter sticks arranged in all possible directions

- This construction works for Minkowski space and so it is reasonable to define asymptotically flat space-times as those for which the scaling down for the metric is possible

- Penrose proved among other things :

  1. If $\bar{g}$ satisfies the Einstein vacuum equations near $\Im$ then $\Im$ is null (This also follows if matter is present near $\Im$ provided the stress-energy tensor is trace-free, e.g., Einstein-Maxwell theory)

  2. $\Im$ consists of two disjoint pieces $\Im^+$ and $\Im^-$ each topologically $R \times S^2$. $\Im^+$ (future null infinity) bounds the physical space-time $\bar{M}$ to the future and $\Im^-$ (past null infinity) bounds $\bar{M}$ to the past

•

The physical space-time $(\bar{M}, \bar{g})$ is conformally embedded in the unphysical space-time (M, g)

•

An isolated system . Future null infinity $\Im^+$ is the half-cone ; the space-time is below it. Time increases upwards, space is horizontal (and one spatial dimension is suppressed) . The central "world tube" represents a region from where gravitational radiation (indicated by wavy lines) is emitted ; these waves leave their profiles on $\Im^+$

- A pure supertranslation has been depicted graphically .

$$\begin{aligned}\bar{u} &= \frac{u+\alpha(\theta,\phi)}{K(\theta,\phi)}\\ \bar{\theta} &= H(\theta,\phi)\\ \bar{\phi} &= I(\theta,\phi)\end{aligned}$$

When $\bar{\theta}=\theta$ and $\bar{\phi}=\phi$ we obtain a pure supertranslation :

$$\bar{u}=u+\alpha(\theta,\phi)$$

- Why are there supertranslations? They come about for a direct physical reason. Imagine a family of observers very far away (near $\Im^{+}$) from the system in question, located at various angular coordinates (θ,ϕ), who synchronize their clocks

- Suppose a gravitational wave passes , and then suppose that they examine their clocks again . They will in general find they have become desynchronized , that is, they have passed from a common retarded time u to one of the form $u+\alpha(\theta,\phi)$. It is to accommodate this physical effect that one must introduce supertranslations

- In special relativity, there are no gravitational waves , this sort of desynchronization cannot occur , and consequently one has no need to introduce the BMS group. But for general relativity where gravitational waves are to be expected, we have no choice

2.4 The theory does not specify the degree of smoothness of the supertranslations : this gives rise to a host of possibilities regarding the allowed topologies

- There is a wide range of choices of "reasonable" topologies, arising from the infinite-dimensional additive supertranslation subgroup $\mathcal{A}$ of "arbitrary" real-valued functions on the Riemann sphere S. Also, the range of choices available depends on the class of functions allowed in $\mathcal{A}$

- The original derivation of B required that the supertranslations are $C^2(S)$

- The original derivation of B was superceded by that of Penrose who gave a derivation of B as that group of *exact* conformal motions of the future (or past) null boundary $\Im^+$ (or $\Im^-$) of conformally compactified weakly asymptotically simple space-times, which preserve “null angles”

- Since truly arbitrary supertranslation functions describe symmetry transformations in Penrose's sense, supertranslations need not have some minimum degree of smoothness. This gives rise to a host of possibilities regarding the allowed topologies:

  1.Cantoni, in his investigation of representations, gave to this $\mathcal{A} = C^2(S)$ the pre-Hilbert topology determined by the area measure of S. Cantoni has shown that, if $L_+^\uparrow$ is given the usual topology, then B_2 is a non-locally compact group in the product topology of $\mathcal{A} \times L_+^\uparrow$

2.McCarthy worked on two choices and proposed a third one :

2.4.1 First choice : B in the Hilbert topology

- McCarthy (Proc.R. Soc. Lond. A **330** , 517-535, 1972, Proc.R. Soc. Lond. A **333**, 317-336, 1973, Proc.R. Soc. Lond. A **335**, 301-311,1973, Proc.R. Soc. Lond. A **358**, 141-171, 1978) in his first study of the representations of B, chose the same topology with the one used by Cantoni, but widened $\mathcal{A} = C^2(S)$ to $\mathcal{A} = L^2(S)$ in order to simplify the treatment. The discreteness result is true with either the Cantoni or McCarthy choices

- Introduce a scalar product into $\mathcal{A}$

$$< \alpha, \beta >= \int_{S^2} \alpha(x)\beta(x)\mathrm{d}\mu(x)$$

where, $x \in S^2$, and, $\alpha, \beta \in \mathcal{A}$

- With this scalar product, $\mathcal{A} = L^2(S)$ becomes a real Hilbert space, and, in the induced metric topology, becomes an (Abelian) topological group

- With a proof similar to Cantoni's, it can be shown that, when $\mathcal{A} = L^2(S)$ is endowed with the aforementioned topology, and $L_+^\uparrow$ is given the usual topology, then B is a non-locally compact group in the product topology of $\mathcal{A} \times L_+^\uparrow$

- The representations are determined by the action of $L_+^\uparrow$ on the topological dual of $\mathcal{A}$, $\mathcal{A}'$. It is a fundamental theorem, the Reisz-Frechet theorem, (the Hilbert space representation theorem), that $\mathcal{A} = L^2(S)$ and $\mathcal{A}'$ are isometrically isomorphic. It is precisely this isomorphism which simplifies the treatment

The three more important results in the Hilbert topology

- B irreducibles, i.e., $B-$elementary entities, describe 'elementary particles'

- In the representation theory of B only compact little groups, hence discrete spins, arise

- There is strong evidence which suggests that a class of IRs of B correspond to ALE gravitational instantons

Open problems

- The main open problem is the physical interpretation of the irreducibles, in particular of those induced from non-connected discrete little groups

- **A crucial remark here which shapes to a great extent the results is the following : The "smoother" you require the functions in $\mathcal{A}$ to be, the "rougher" you can allow the generalized functions in $\mathcal{A}'$ to be.**

2.4.2 Second choice : B in the nuclear topology

- McCarthy (Proc.R. Soc. Lond. A **343**, 489-523, 1975, Proc.R. Soc. Lond. A **351**, 55-70, 1976) in his subsequent study of the representations of B chose $\mathcal{A} = C^{\infty}(S)$

- This choice by no means determines the topology uniquely. However, a natural choice for vector spaces whose elements are "smooth" in some sense, e.g., for the set of smooth functions on a compact manifold, is the nuclear topology

- The nuclear topology is determined by the following notion of convergence : a sequence of functions a_k on the sphere is said to converge to zero, if and only if, the functions a_k together with all their derivatives of all orders, converge uniformly to zero over the sphere

- The expectation is that in a finer topology, the nuclear is finer than the Hilbert, more representations become continuous, and some have non-compact little groups, hence, possibly, continuous spins

The two more important results in the nuclear topology

- It appears that the extra representations in the nuclear topology (which may have continuous spins) describe 'scattering' states, and the remaining ones are identified with bound states, corresponding to elementary particles

- The particles appropriate to B have discrete spins irrespective of the choice of topology, Hilbert or nuclear

Open problems

- We do not know if the inducing construction is exhaustive

- Some of the irreducibles are induced from infinite discrete subgroups of $SL(2, C)$, which are far from being known!

2.4.3 Third choice : $\mathcal{A}'$ becomes enlarged to the space of real hyperfunctions $\mathcal{B}(S)$ on the Riemann sphere S

- If $\mathcal{A}$ is taken to consist of real analytic C^ω functions rather than C^∞ functions then, with an appropriately fine topology, the dual space $\mathcal{A}'$ becomes enlarged to the space of real hyperfunctions $\mathcal{B}(S)$ on the Riemann sphere S

- The space of hyperfunctions is larger than the space of distributions on S

- This new topology on $\mathcal{A} = C^\omega(S)$, and the associated hyperfunctional supermomenta, are probably more 'physical' than the nuclear topology and the associated distributional supermomenta

- It seems that the category of hyperfunctions is more appropriate than that of distributions for discussing S-matrix theory

- Since the representation theory of B in nuclear topology strongly suggests that the physical situations being analyzed are related to scattering problems, the present discussion of $\mathcal{A} = C^{\omega}(S)$ seems to imply that one should really consider hyperfunctional supermomenta, or more generally, hyperfunctional solutions to Einstein's equations.

- Interestingly, quantum field theories in which fields are smeared by hyperfunctions show a non local behaviour and the density of states can have a non polynomial growth.

- This might in principle allow to recover bulk locality, in the exploration of the holographic principle in asymptotically flat spacetimes via the BMS group, although one should consider hyperfunctional solutions to the Einstein equations

- More remarkably, if one assumes that the high energy behaviour of the density of states in the bulk is dominated by black holes, the exponential growth of states which suggests an intrinsic degree of non locality, might be explained by working with hyperfunctions

2.5 A more familiar situation where changing the topology alters the physics

- Zeeman has proposed a topology on Minkowski space, hence on the Poincare group P, which is finer than the usual one, but physically plausible because it is adapted to the null-cone structure. In Zeeman's finer topology, one would expect many (unknown) representations of P in addition to the usual ones

2.6 Research Programme

2.6.1 Following through Wigner's programme with B replacing P

- In 1939 Wigner published a remarkable paper which laid the foundations of special relativistic quantum mechanics

- In this paper
  1. The the set of states of a quantum system was identified with the projective space $P(H)$ of complex straight lines, through the origin, of a complex Hilbert space H
  2. Moreover, it was postulated, that all transition probabilities are invariant under all Poincare transformations

- Note : For applications to quantum mechanics , one deals with projective representations , rather than true representations, and these may all be found by determining the true representations of the universal covering group $\mathcal{P}$. $\mathcal{P}$ is defined by replacing $L_+^\uparrow$ by $SL(2,C)$.

- Wigner's main result : He gave a complete classification of all relativistic invariant systems in terms of irreducible unitary representations (IRs) of $\mathcal{P}$ in H

- *First theoretical "definition" of elementary particle* : These IRs were, in turn, identified with elementary particles , and shown to be parameterized by mass and spin

- Wigner's work describes , explicitly, the set of all possible solutions of all possible (specially) relativistic wave equations , without having to find or solve the equations ! These solution sets are, in fact, precisely the IRs of $\mathcal{P}$

- B was discovered by Bondi and coworkers for axisymmetric systems , and by Sachs (Proc. R. Soc. Lond. A **270,** 103-126 , 1962) for general systems , and is the best candidate for the universal symmetry group of general relativity

- As such it quickly attracted attention as an approach to quantum gravity , or the problem of "internal symmmetries " (Sachs Phys. Rev.**128** , 2851-2864,1962, Komar Phys. Rev. Lett. **15** , 76-78, 1965 , Newman Nature, **206** ,811, 1965) . With these motivations a study of the IRs of B was started by Sachs Phys. Rev.**128** , 2851-2864,1962 , and taken further by Cantoni JMP, **8** , 1700-1706, 1967

- Wigner's work for special relativity , and the universal property of B , make it reasonable to attempt to lay a similarly firm foundation for quantum gravity, by following through the analogue of Wigner's programme with B replacing P

- Some years ago McCarthy Proc.R. Soc. Lond. A **333** , 317-336, 1973, Proc.R. Soc. Lond. A **343**, 489-523, 1975, Proc.R. Soc. Lond. A **348**, 141-171, 1978, constructed the IRs of B for exactly this purpose . This work was based on Mackey's pioneering work on group representations
- The role of IRs of B is much less well understood than the role of IRs of P
- To make this role better understood and to relate the group theoretical approach more closely to other approaches to quantum gravity where complexified or euclidean versions of general relativity are frequently considered McCarthy Phil. Trans. R. Soc. Lond. A. **338** , 271-299 ,1992 constructed analogues of B for these versions of the theory , and a variety of further ones, either real in any signature, or complex

2.6.2 Results on the representation theory of B in the Hilbert topology, Comparison with the representation theory of P

- The two more important results which have been derived so far from the representation theory of B are :
  1. All the B-elementary entities carry discrete spin (McCarthy Proc. R. Soc. Lond. A. **333** , 317 ,1973)
  2. There is strong evidence which suggests that a class of IRs of B correspond to ALE gravitational instantons
- It is as though the presence of gravity obstructs the unphysical continuous spins of special relativity. That is, gravity gives a possible explanation for the observed discreteness of elementary particle spins

- There are two striking differences concerning the little groups of P and B
- The little groups of P are

$$SU(2), \quad \Delta, \quad SU(1,1)$$

where

1. SU(2) is the subgroup of $SL(2,C)$ consisting of unitary matrices and is the double cover of $SO(3)$. It is compact
2. Δ is the subgroup of $SL(2,C)$ consisting of upper triangular matrices whose diagonal elements have unit modulus and is the double cover of the group $E(2)$ of Euclidean motions of the plane. It is not compact
3. SU(1,1) is the subgroup of $SL(2,C)$ consisting of pseudo-unitary matrices for a signature (+,-) and is the double cover of the 2+1 Lorentz group $SO(2,1)$. It is not compact

- The little groups of B are

  1. **Connected**
     - $SU(2)$
     - Γ where, Γ is the double cover of $SO(2)$
  2. **Non-Connected**
     - $\Theta = \Gamma R_2$, where $R_2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$
     - $\bar{C}_n$, where C_n is the cyclic group of order n , of order 2n
     - $\bar{D}_n$, where D_n is the dihedral group of order $2n$, of order 4n
     - $\bar{T}$, where T is the symmetry group of the tetrahedron, of order 24
     - $\bar{O}$,where O is the symmetry group of the cube, of order 48
     - $\bar{I}$,where I is the symmetry group of the icosahedron, of order 120

- First striking difference : *Some* of the little groups of P are compact , whereas *ALL* the little groups of B are compact

- Compact little groups always give discrete spins , the non-compact ones also give continuous spins and this is precisely the case for P . Some of the representations of Δ correspond to $m^2 = 0$ particles with discrete spins, whereas, the rest, correspond to $m^2 = 0$ particles with continuous spins

- However, the little groups for B are always compact and this means that the IRs of B necessarily have only discrete spins

- The little groups of P are all of infinite order (they are all 3-dimensional connected Lie groups), some of the non-connected little groups of B are finite

- The finite little groups of B are precisely the (double covering groups of the) symmetry groups of the regular polygons and polyedra in ordinary Euclidean 3-space
- These are the double covering groups of
  1. The cyclic groups C_n , $n \geq 1$
  2. The dihedral groups D_n , $n \geq 1$
  3. The group T of the tetrahedron
  4. The group O of the cube (which coincides with that of the octahedron)
  5. The group I of the icosahedron (coincides with that of the dodecahedron)

– **The possibility of non-connected (or even discrete) little groups, is unfamiliar in physics, and, in this context, is a feature peculiar to the infinite dimensionality of supermomentum space, i.e., for $\mathcal{A}'$, (which allows a lot of freedom for invariant vectors). It is essential to take this feature into account for B, and for its generalizations, otherwise most of the interesting information is lost**

2.6.3 Comparison of the representation theory of B in the nuclear topology with the representation theory of P

– There are precisely four connected little groups of B in the nuclear topology. They are Γ, $SU(2)$, Δ, and $SL(2, R)$. Thus we have the interesting result that, in the nuclear topology, the connected little groups of B are those of the Poincare group, and precisely one extra one, Γ

– Many additional non-connected little groups appear for B in the nuclear topology. Non-connected subgroups associated with (i.e. with identity component) $SU(2)$, Δ and Ω do not occur as little groups for B in the nuclear topology. Non-connected subgroups associated with every other connected subgroup do occur as little groups for B

2.6.4 Possible link with instantons

- The much later appearance (Kronheimer J.Dif. Geom. **(29)** , 665 , 1989, and, J.Dif. Geom. **(29)** , 685 , 1989) of precisely the same complex linear IRs of precisely the same finite groups (and not just the groups themselves) suggests a connection with the IRs of B

- Kronheimer gave a complete classification of ALE (Asymptotically Locally Euclidean) gravitational instantons. ALE gravitational instantons are a class of asymptotically flat solutions (or at least,locally so) of the self-dual Euclidean Einstein equations
- Locally : Outside some compact region the metric approaches the flat (Euclidean) metric on $(S^3/T) \times R$, where, T is a finite group of isometries acting freely on S^3

- Kronheimer found that the parametrization of the instanton solution spaces (moduli spaces) intimately involves the complex linear IRs of the finite symmetry groups of the regular polygons and polyedra

- Kronheimer did not give explicit expressions for the metrics of the ALE spaces

- The two approaches to quantum gravity (via BMS IRS and via euclidean instantons) could hardly be more different

- The former starts with an infinite-dimensional group (the symmetry group B of the theory) and a Hilbert space only (no equation is postulated or solved)

- The latter starts with the self-dual Einstein equations (no symmetry groups or transformation properties are assumed) and constructs the moduli spaces of solutions of these equations

- Either the simultaneous appearance of precisely the same IRs of precisely the same finite groups is merely a coincidence , or the two approaches in similar contexts (asymptotically flat space-times , quantum gravity) are deeply related
- Thus , it is essential first to relate the two approaches as closely as possible. Gravitational instantons appear in a complexified or euclidean version of General Relativity , but BMS IRs have only been investigated in real Lorentzian space-times. Are there complex or euclidean analogues of B ?

2.7 Generalizations

- If so , how are these new groups related to each other and to B? What are the IRs of these new groups? Quite apart from the possible link with instantons , these questions are of much independent interest

- With this motivation , McCarthy defined (Phil. Trans. R. Soc. Lond. A **338** , 271, 1992) all possible generalizations of B , for complex space-times, space-times of any signature and obtained 42 groups

- These 42 groups have the general structure

$$\mathcal{G} = C^{\infty}(A, R)\circledS_T H$$

,where,

  – A denotes a compact space

  – $C^{\infty}(A, R) \trianglelefteq \mathcal{G}$ are the supertranslations ; they are infinitely -differential functions defined on A

- Some examples are

$$\begin{aligned} CB &= C^{\infty}(S^2 \times S^2, R)\circledS_T \\ &\quad (SL(2, C) \times SL(2, C)) \end{aligned}$$

$$EB = C^{\infty}(S^3, R)\circledS_T(SU(2) \times SU(2))$$

$$\begin{aligned} B(2,2) &= C^{\infty}(S^1 \times S^1, R)\circledS_T \\ &\quad (SL(2, R) \times SL(2, R)) \\ H(2,2) &= C^{\infty}(P_1(R) \times P_1(R), R)\circledS_T \\ &\quad (SL(2, R) \times SL(2, R)) \\ B &= C^{\infty}(S^2, R)\circledS_T SL(2, C) \end{aligned}$$

2.8 Wigner-Mackey's theory in a nutshell

- To construct the IRs of the aforementioned 42 groups we use Wigner-Mackey's representation theory (Wigner Ann.Math. **40** 149, 1939)
- Wigner-Mackey's theory finds IRs of the type
$$\mathcal{B} = \mathcal{A} \textcircled{s}_T G$$
,where,
  1. $\mathcal{A}$ is abelian normal subgroup of $\mathcal{B}$
  2. T is an homomorphism
  $$T : G \to Aut(\mathcal{A})$$
  3. $\mathcal{A}$ and G are topological groups. In the product topology of $\mathcal{A} \times G$, $\mathcal{B}$ then becomes a topological group. It is assumed that it becomes a separable locally compact topological group

2.8.1 Basics

- **Characters - The Dual Group** The IRs of $\mathcal{A}$ are one - dimensional . They can be given the structure of an abelian group $\hat{\mathcal{A}}$, the dual group of $\mathcal{A}$, with group operation given by
$$(\chi_1\chi_2)(\alpha) = \chi_1(\alpha)\chi_2(\alpha)$$
, where, $\chi_1, \chi_2 \in \hat{\mathcal{A}}$, $\alpha \in \mathcal{A}$
- **Form of the characters** Typically one introduces a scalar product into $\mathcal{A}$, defined by
$$< \alpha, \beta >= \int_{S^2} \alpha(x)\beta(x)\mathrm{d}\mu(x)$$
, where, $x \in S^2$, and, $\alpha, \beta \in \mathcal{A}$. With this scalar product $\mathcal{A}$ becomes a real Hilbert space in the usual way , and,
$$\chi(\alpha) = e^{i<\phi,\alpha>}$$
, where, $\phi, \alpha \in \mathcal{A}$, and, $< \phi, \alpha >$ is the aforementioned scalar product

- $\mathcal{A}$ **is also a real vector space**

$$(\chi_1 + \chi_2)(\alpha) := \chi_1(\alpha) + \chi_2(\alpha)$$
$$(\lambda \circ \chi)(\alpha) := e^{i<\lambda\phi,\alpha>}, \text{ where, } \lambda \in R$$

- **Dual Action** The action T of G on $\mathcal{A}$ induces a dual action $\hat{T}$ of G on $\hat{\mathcal{A}}$ defined by

$$(\hat{T}(g)\chi)(\alpha) := \chi(T(g^{-1})\alpha)$$

It is precisely this dual action which determines the structure of the IRs of $\mathcal{B}$

- **Little Groups** L_χ For a *given* χ the largest subgroup L_χ of G which leaves χ fixed is called the little group of χ, i.e.,

$$L_\chi = \{g \in G | \hat{T}(g)\chi = \chi\}$$

- **The Orbit of χ, denoted by G_χ**

$$G_\chi = \{g\chi | g \in G\}$$

$\hat{\mathcal{A}}$ is partitioned by the orbits G_χ

2.8.2 Constructing the Hilbert space $\mathcal{H}_\nu$ in which the IRs of $\mathcal{B}$ are materialized

- **There is a natural bijection**

$$G/L_\chi \quad \longleftrightarrow \quad G\chi$$
$$, \text{ given by},$$
$$gL_\chi \quad \longleftrightarrow \quad g\chi$$

- The coset space G/L_χ has a unique class of quasi-invariant measures for the G-action ; let ν be one of these

- Let U be a continuous irreducible of the little group L_χ on a Hilbert space $\mathcal{D}$

- Let $\mathcal{H}_\nu$ be the space of functions $\psi : G \to \mathcal{D}$ which satisfy the conditions

  1. $\psi(gh) = U(h^{-1})\psi(g)$
  2. $\int_{G\chi} < \psi(p), \psi(p) > \mathrm{d}\nu(p) < \infty$

  ,where, $h \in L_\chi, g \in G$, and, where, the scalar product under the integral sign is that of $\mathcal{D}$

- **Remark** The integrand is expressed as a function on $G\chi \longleftrightarrow G/L_\chi$ since, in view of 1, the integrand is constant on the cosets in G/L_χ

- $\mathcal{H}_\nu$ is turned into a Hilbert space by introducing the scalar product

$$< \psi_1, \psi_2 >= \int_{G\chi} < \psi_1(p), \psi_2(p) > \mathrm{d}\nu(p)$$

- **Definition of the representation of $\mathcal{B}$ on $\mathcal{H}_\nu$**
  Define an action of $\mathcal{B}$ on $\mathcal{H}_\nu$ by
  $$\begin{aligned}(g_o\psi)(g) &= \psi(g_o^{-1}g)\\ (\alpha\psi)(g) &= [(\hat{T}(g)\chi)(\alpha)]\psi(g),\end{aligned}$$
  where, $g, g_o \in G$, and $\alpha \in \mathcal{A}$. This action gives a unitary representation of $\mathcal{B}$ on $\mathcal{H}_\nu$ which is continuous whenever U is.

- **This is precisely the representation of $\mathcal{B}$ induced from χ and the irreducible representation U of the little group L_χ**

2.8.3 Why little groups might change when topology is modified

- The induced representations are associated with the existence of invariant characters, i.e., of elements in $\mathcal{A}'$, the topological dual of $\mathcal{A}$, (supermomenta in B, generalizations of the Poincare momenta), which are left invariant by the action of some subgroup (little group) of $SL(2, C)$

- Little groups are then connected, via $\mathcal{A}'$, to the topology of $\mathcal{A}$

- E.g., a refinement of the topology of $\mathcal{A}$ may broaden $\mathcal{A}'$, and therefore, new invariant elements with associated little groups may come into existence

- This is precisely what happens when from the Hilbert topology we pass to the nuclear topology

2.8.4 Mackey's Theorems

- **First Theorem :** Given the topological restrictions on $\mathcal{A} = \mathcal{A}\circledS_T G$ (separability and local compactness), *any* representation of $\mathcal{B}$, constructed by the method above , *is irreducible if the representation U of L_χ on $\mathcal{D}$ is irreducible.*
- **Main Conclusion : An irreducible representation of $\mathcal{B}$ is obtained for each $\chi \in \hat{\mathcal{A}}$ and each irreducible representation U of L_χ**
- **Second Theorem :** If $\mathcal{B} = \mathcal{A}\circledS_T G$ is a regular semi-direct product (i.e., $\hat{\mathcal{A}}$ contains a Borel subset which meets each orbit $G\chi$ in $\hat{\mathcal{A}}$ in just one point - **the G- action is not too pathological when the G-orbits can be enumerated in some way**) then **all** of its irreducibles representations can be obtained in this way

2.8.5 Problems ,and, McCarthy's and Piard's resolutions

- **First Problem : The 42 groups are not locally compact in the Hilbert topology.** The First Theorem dealing with the irreducibility of the induced representations obtained by the above construction no longer applies. However, for the 42 groups in question the induced representations obtained above *are* irreducible. The proof follows closely the one given in McCarthy & Crampin Proc. R. Soc. Lond. A **335** , 331 for the case of the original BMS group B.

- **Second Problem : Mackey's regularity condition is** *not* suitable for infinite-dimensional groups.
  When $\mathcal{B} = \mathcal{A}ⓢ_T G$ is infinite-dimensional, and when $\mathcal{A}$ is endowed with the Hilbert topology, Piard, was able to show, (Rep. Math. Phys. **11,** 259, 1977, and, Rep. Math. Phys. **11,** 279 , 1977) that Mackey's regularity condition can be replaced by the following two regularity conditions

  1. The orbits are open relative to their strong closure,i.e., *for every orbit G_χ there is an open set O such that*
  $$G_\chi = O \cap \tilde{G}_\chi,$$
  where , $\tilde{G}_\chi$ is the strong closure of G_χ
  2. Two distinct *minimal invariant cones* have distinct weak closures , where, the *minimal invariant cone generated by* $\chi \in \hat{\mathcal{A}}$ is defined by
  $\mathcal{C}_\chi = \{\lambda(g\chi)\ ,\ g \in G, \chi \in \hat{\mathcal{A}}, \text{and}, \lambda \in R^+\}$

- It is clear that $\mathcal{C}_\chi$ contains all the orbits $G_{\lambda\chi}$, $\lambda > 0$

- The first condition "tames" the action of G on $\hat{\mathcal{A}}$. In fact Piard showed that if the first condition holds then the minimal invariant cones $\mathcal{C}_\chi$ satisfy one of the following three properties
  1. The half-line $\lambda\chi$ ($\lambda \geqslant 0$) intersects each orbit of $\mathcal{C}_\chi$ at one and only one point
  2. There exists an interval $[\alpha, \beta\alpha)$, $\beta > 0$, on the half-line $\lambda\alpha$ ($\lambda > 0$) which intersects each orbit of $\mathcal{C}_\chi$, except $\{0\}$, at one and only one point
  3. The half-line $\lambda\chi$ ($\lambda > 0$) is included in the orbit G_χ

- The synergy of the two conditions assures, as was shown by Piard, that every (cylindrical ergodic) measure is concentrated on an orbit, i.e., there exists an orbit the complement of which is a null set. This is the well - known main problem even in the finite-dimensional case : It is only then that *all* the IRs of $\mathcal{B}$ are induced by representations of the little groups L_χ

- Piard's condition holds in the case of the original BMS group B when this is endowed with the Hilbert topology. It is not immediate that it holds for the remaining 41 groups . This should be checked case by case

2.9 Summary of the Program (in the Hilbert topology)

- **Find the little groups L_χ and the associated invariant spaces $L^2(L_\chi)$**

- **Check that Piard' regularity condition holds**

- **This has been done for $B(2,2)$ and $H(2,2)$**

When B and its variants are endowed with the nuclear topology simply we do not know if the inducing constru–ction is exhaustive

2.10 B(2,2)

-
$$B(2,2) = C^\infty(S^1 \times S^1, R)\circledS_T \\ (SL(2,R) \times SL(2,R)),$$
where, $\alpha(m,n) \in S^1 \times S^1$ are "even" functions,i.e.,
$$\alpha(m,n) = \alpha(-m,-n),$$
where, $m = \frac{x}{|x|}, x = (x_1, x_2) \in R^2 - \{0\}$, and , similarly, $n = \frac{y}{|y|}, y = (y_1, y_2) \in R^2 - \{0\}$

- $B(2,2)$ is the BMS group appropriate to the "ultrahyperbolic" signature and asymptotic flatness in null directions

- The ultrahyperbolic version of Minkowski space, sometimes written $R^{2,2}$, is just R^4 - the vector space of row vectors with 4 real components - with scalar product $x \circ y$ between x and y given by

$$x \circ y = x^0 y^0 + x^2 y^2 - x^1 y^1 - x^3 y^3$$

- Typically a pre-Hilbert space structure is given to $C^\infty(S^1 \times S^1, R)$ by defining a scalar product

$$< \alpha, \beta >= \int_{S^1 \times S^1} \alpha(m,n)\beta(m,n) \mathrm{d}\lambda(m,n)$$

, where, $\alpha, \beta \in C^\infty(S^1 \times S^1, R)$. As for B, it is convenient to complete the space with respect to the norm defined by the scalar product. So we now deal with the complete group

$$L^2_e(T^2, R) \circledS_T G^2,$$

where, $T^2 = S^1 \times S^1$, and, $G = SL(2, R)$

- The little groups L_ϕ and the associated invariant spaces $L^2_e(L_\phi)$ are the following

Little groups L_ϕ and the corresponding invariant spaces $L^2_e(L_\phi)$

	L_ϕ	$L^2_e(L_\phi)$
1.	$\mathrm{SO}(2) \times \mathrm{SO}(2) = (R(\vartheta), R(\varphi))$	$\widetilde{\phi}(\rho,\sigma) = c, \quad$ some $\quad c \in R$
2.	$\mathrm{C_N} \times \mathrm{SO}(2) = \left(R\left(\frac{2\pi}{\mathrm{N}}i\right), R(\varphi)\right)$, where N is even.	$\widetilde{\phi}(\rho,\sigma) = g(\rho)$ $g(\rho)$ is periodic of period $\frac{2\pi}{\mathrm{N}}$ $\int_0^{\frac{2\pi}{\mathrm{N}}} (g(\rho))^2 \,\mathrm{d}\rho < +\infty$
3.	$\mathrm{SO}(2) \times \mathrm{C_N} = \left(R(\vartheta), R\left(\frac{2\pi}{\mathrm{N}}i\right)\right)$, where N is even.	$\widetilde{\phi}(\rho,\sigma) = l(\sigma)$ $l(\sigma)$ is periodic of period $\frac{2\pi}{\mathrm{N}}$ $\int_0^{\frac{2\pi}{\mathrm{N}}} (l(\sigma))^2 \,\mathrm{d}\rho < +\infty$
4.	H(N,p,q)$= \left(R(\mathrm{p}\vartheta), R\left(\mathrm{q}\vartheta + \frac{2\pi}{\mathrm{N}}i\right)\right)$, where, either, both p and q are odd, $\mathrm{p/N} = \mathrm{p'/N'}$, where $\mathrm{p', N'}$ are coprime, or, p and q have opposite parity, $\mathrm{p/N} = \mathrm{p'/N'}$, where $\mathrm{p', N'}$ are coprime, and, $\mathrm{N'}$ is even.	$\widetilde{\phi}(\rho,\sigma) = f(\mathrm{p}\sigma - \mathrm{q}\rho) \equiv f(\widehat{\sigma})$ $f(\widehat{\sigma})$ is periodic of period $\frac{2\pi}{\mathrm{N'}}$ $\int_0^{\frac{2\pi}{\mathrm{N'}}} (f(\widehat{\sigma}))^2 \,\mathrm{d}\widehat{\sigma} < +\infty$
5.	Subgroups $\mathcal{C}$ of $\mathrm{C_n} \times \mathrm{C_m}$ which contain the element $(R(\pi), R(\pi)) = (-\mathrm{I}, -\mathrm{I})$. Both n and m are finite and even.	$\widetilde{\phi}(\rho,\sigma) = 0 \;\mid\; (\rho,\sigma) \notin \mathrm{E}_{\mathcal{C}}$ $\int_{\mathrm{E}_{\mathcal{C}}} (\widetilde{\phi}(\rho,\sigma))^2 d\rho \wedge d\sigma < +\infty$

where, $\mathrm{E}_{\mathcal{C}}$ is an elementary domain for the $\mathcal{C}$- action on T^2

- Regarding the structure of the little groups of $B(2,2)$ and $H(2,2)$ we note that
  1. All the little groups are compact
  2. There is one two-dimensional little group
  3. The one-dimensional little groups form an unexpected family of continuous/discrete groups with many connected components
  4. There are finite little groups
  5. The finite little groups involve subgroups of direct products of the symmetry groups of the regular *polygons* only ; the regular polyhedra do not appear at all hear

2.11 Open Questions

- Try to understand the physical meaning of the IRs of B and of its variants. In particular, of the IRs which are induced from non-connected, and even discrete, little groups. Possibly the construction of the B- invariant wave equations is a good starting point

- Establish the connection with (ALE) instantons

- Explore if the holographic principle of t'Hooft and Susskind is realized in asymptotically-flat space-times. Two routes have been pursued : In one of them the holographic boundary is the future null infinity and the QFT at the boundary is 'B-invariant', in the other, the symmetry algebra of asymptotically flat space-times at null infinity in 4 dimensions is taken to be the semi-direct sum of supertranslations with infinitesimal *local* conformal transformations and not, as usually done, with the Lorentz algebra. Try to put the holoscreen at spatial infinity, since then the analogy with AdS/CFT correspondence is better