Precise analytic treatment of Kerr and Kerr-(anti) de Sitter black holes as gravitational lenses.

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Abstract

The null geodesic equations that describe motion of photons in Kerr spacetime are solved exactly in the presence of the cosmological constant Λ. The exact solution for the deflection angle for generic light orbits (i.e.non-polar,non-equatorial) as well as for equatorial, is calculated in terms of the generalized hypergeometric functions of Appell and Lauricella.

We then consider the more involved issue in which the black hole acts as a 'gravitational lens'. The constructed Kerr black hole gravitational lens geometry consists of an observer and a source located far away and placed at arbitrary inclination with respect to black hole's equatorial plane. The resulting lens equations are solved elegantly in terms of Appell-Lauricella hypergeometric functions. In this framework, the magnification factors for generic orbits which include equatorial and polar orbits are calculated in closed analytic form for the first time. The exercise is repeated with the appropriate modifications for the case of non-zero cosmological constant.

1 Some history.

The gravitational bending of light (and the associate phenomenon of gravitational lensing) has been instrumental in unravelling the nature of the gravitational field and its cosmological implications:

- Back in 1801, Johann Georg von Soldner using Newtonian mechanics and assuming a corpuscular theory for light derived a value for the deflection angle in the Sun's gravitational field:$\Delta\phi^S = 0''.875$.
- Later, Einstein using the equations of General Relativity derived a value of $\sim 1''.75$ which is consistent with the findings of Eddington's solar eclipse experiment and subsequent measurements.
- Recent progress: the closed form (strong-field) solution has been derived for the deflection angle of an equatorial ray in the Kerr gravitational field (spinning black hole, rotating mass). The results exhibited clearly the strong dependence of the gravitational bending of light on the spin of the black hole (frame-dragging effects) for small values of the impact parameter:

Kraniotis,CQG22(2005)4391–4424

2 Null geodesics in Kerr and Kerr-de Sitter spacetime.

Taking into account the contribution from the cosmological constant, Λ, the generalization of the Kerr solution is described by the Kerr-de Sitter metric element which in Boyer-Lindquist (BL) coordinates is given by [Stuchlik Calvani,Gen.R.Grav23 (1991),Demianski(1973)ActaAstron.23, Carter,Com.M.P.(1968)]:

$$\begin{aligned} \mathrm{d}s^2 = & \frac{\Delta_r}{\Xi^2\rho^2}(c\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 - \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\ & - \frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}(ac\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2 \end{aligned} \tag{1}$$

$$\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\theta, \ \Xi := 1 + \frac{a^2\Lambda}{3} \tag{2}$$

$$\Delta_r := \left(1 - \frac{\Lambda}{3}r^2\right)\left(r^2+a^2\right) - 2\frac{GM}{c^2}r \tag{3}$$

(rdel)

We denote by a the rotation (Kerr) parameter and M denotes the mass of the spinning black hole.

The relevant null geodesic differential equations for the calculation of the gravitational lensing effects (lens-equation) and for the calculation of the deflection angle are:

Kraniotis,CQG22(2005)4391–4424

$$\boxed{\int^r \frac{\mathrm{d}r}{\sqrt{R}} = \pm \int^\theta \frac{\mathrm{d}\theta}{\sqrt{\Theta}}} \tag{4}$$

and

$$\boxed{\Delta\phi = \int \mathrm{d}\phi = \int^\theta -\frac{\Xi^2}{\Delta_\theta \sin^2\theta}\frac{(a\sin^2\theta - \Phi)\mathrm{d}\theta}{\sqrt{\Theta}} + \int^r \frac{a\Xi^2}{\Delta_r}[(r^2+a^2) - a\Phi]\frac{\mathrm{d}r}{\sqrt{R}}} \tag{5}$$

(lens),(ray)

where

$$R := \left\{\Xi^2\left[(r^2+a^2) - a\Phi\right]^2 - \Delta_r\left[\Xi^2(\Phi - a)^2 + \mathcal{Q}\right]\right\} \tag{6}$$

and

$$\Theta := \left\{[\mathcal{Q} + (\Phi - a)^2\Xi^2]\Delta_\theta - \frac{\Xi^2(a\sin^2\theta - \Phi)^2}{\sin^2\theta}\right\} \tag{7}$$

We also derive the equation related to time-delay:

$$ct = \int^r \frac{\Xi^2(r^2+a^2)\left[(r^2+a^2) - \Phi a\right]}{\Delta_r\sqrt{R}} - \int^\theta \frac{a\Xi^2(a\sin^2\theta - \Phi)}{\Delta_\theta\sqrt{\Theta}} \tag{8}$$

The parameters Φ, $\mathcal{Q}$ are associated to the first integrals of motion. The former is the impact parameter and the latter is related to the hidden first integral (due to the separation of variables in the corresponding Hamilton-Jacobi PDE).

We now turn our attention to the issue of treating the Kerr and Kerr-de Sitter black holes as gravitational lenses, construct the resulting geometry and lens equations and solve the latter in closed analytic form. In addition, we shall derive for the first time the solutions in closed analytic form for the magnification factors.

Previous efforts on the issue of gravitational lensing from a Kerr black hole were concentrated on various approximations and numerical techniques Bray I., *Phys.Rev.*D. **34** (1986) 367;S.E. Vazquez and E.P. Esteban, Nuov.Cim. 119B(2004) 489, M. Sereno and F. De Luca, *Phys.Rev.*D. **74** (2006) 123009, arXiv:astro-ph/0609435v2; V. Bozza, F. De Luca and G. Scarpetta, *Phys.Rev.*D. **74** (2006) 063001; V. Bozza, *Phys.Rev.*D. **78** (2008) 063014.

3 The Kerr black hole

Assume without loss of generality that the observer's position is at $(r_O, \theta_O, 0)$. Likewise, for the source we have (r_S, θ_S, ϕ_S) . In the observer's reference frame, an incoming light ray is described by a parametric curve $x(r), y(r), z(r)$, where $r^2 = x^2 + y^2 + z^2$. For large r this the usual radial BL coordinate.

At the location of the observer, the tangent vector to the parametric curve is given by: $\boxed{(\mathrm{d}x/\mathrm{d}r)|_{r_O}\widehat{\mathbf{x}} + (\mathrm{d}y/\mathrm{d}r)|_{r_O}\widehat{\mathbf{y}} + (\mathrm{d}z/\mathrm{d}r)|_{r_O}\widehat{\mathbf{z}}}$. This vector describes a straight line which intersects the (α, β) plane or *observer's image plane* as it is usually called at (α_i, β_i). The point (α_i, β_i) is the point $(-\beta_i \cos\theta_O, \alpha_i, \beta_i \cos\theta_O)$ in the (x, y, z) system.

Our purpose now is to relate the α_i, β_i variables to the first integrals of motion $\Phi, \mathcal{Q}$. For this we need to use the equation of straight line in space. A straight line can be defined from a point $P_1(x_1, y_1, z_1)$ on it and a vector $\overline{\epsilon}(\epsilon_{1,}\epsilon_{2,}\epsilon_3)$ parallel to it. The analytic equations of straight line are then:

$$\frac{x - x_1}{\epsilon_1} = \frac{y - y_1}{\epsilon_2} = \frac{z - z_1}{\epsilon_3} \tag{9}$$

Applying (9) we derive the equations:

$$\frac{-\beta_i \cos\theta_O - r_O \sin\theta_O}{r_O \cos\theta_O \frac{d\theta}{dr}|_{r=r_0} + \sin\theta_O} = \frac{\alpha_i}{r_O \sin\theta_O \frac{d\phi}{dr}|_{r=r_O}} = \frac{\beta_i \cos\theta_O - r_O \cos\theta_O}{\cos\theta_O - r_O \sin\theta_O \frac{d\theta}{dr}|_{r=r_O}} \tag{10}$$

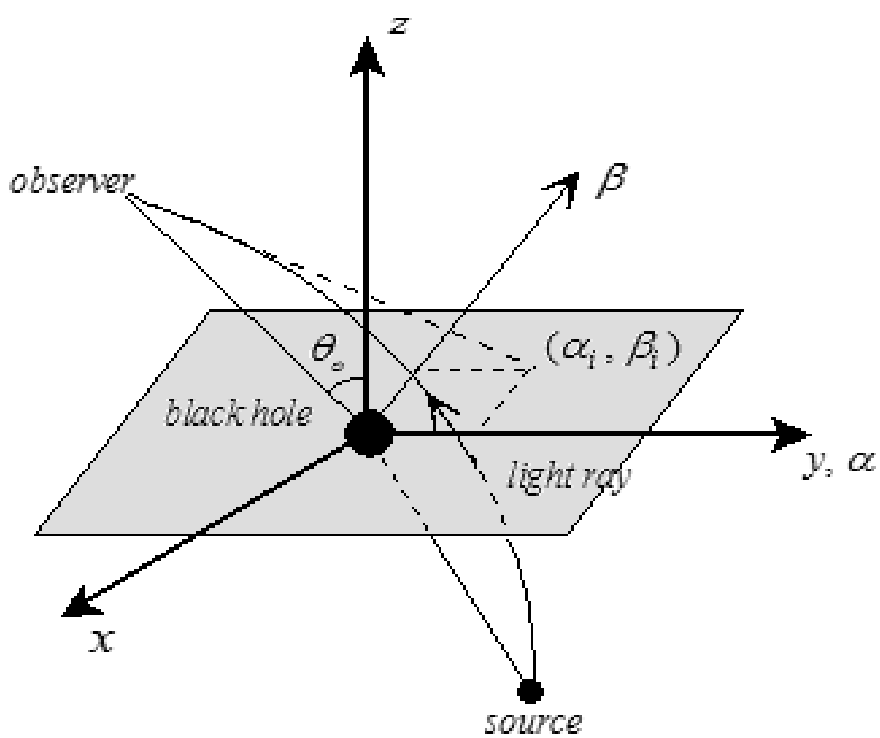
Solving for α_i, β_i we obtain the equations:

$$\alpha_i = -r_O^2 \sin\theta_O \frac{d\phi}{dr}|_{r=r_O} \tag{11}$$

$$\beta_i = r_O^2 \frac{d\theta}{dr}|_{r=r_O} \tag{12}$$

Now we have from the null geodesics that:

$$\frac{d\theta}{dr}|_{r=r_O} = \frac{\Theta(\theta_O)^{1/2}}{R(r_O)^{1/2}} \tag{13}$$


z
observer
β
θ_o
(α_i, β_i)
black hole
y, α
light ray
x
source

and

$$\frac{d\phi}{dr}|_{r=r_O} = \frac{\Phi}{\sqrt[2]{R(r_O)}}\frac{1}{\sin^2(\theta_O)} + \frac{2aGM\frac{r_O}{c^2} - a^2\Phi}{r_O^2\left[1+\frac{a^2}{r_O^2}-\frac{2GM}{r_O c^2}\right]}\frac{1}{\sqrt[2]{R(r_O)}} \tag{14}$$

Using eqns(13),(14) and assuming large observer's distance r_O(i.e. $r_O \longrightarrow \infty$) we derive simplified expressions relating the coordinates (α_i, β_i) on the observer's image plane to the integrals of motion:

$$\boxed{\Phi \simeq -\alpha_i \sin\theta_O} \tag{15}$$

$$\boxed{\mathcal{Q} \simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2(\theta_O)} \tag{16}$$

(IP)
We can also express the position of the source on the observer's sky in terms of its coordinates (r_S, θ_S, ϕ_S) and the observer coordinates. Indeed, the equation for a straight line can be determined by two points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{17}$$

Thus applying the above formula for the straight line connecting the observer and the source yields the equations:

$$\boxed{\alpha_S = \frac{r_O r_S \sin\theta_S \sin\phi_S}{r_O - r_S(\cos\theta_S\cos\theta_O + \sin\theta_O\sin\theta_S\cos\phi_S)}} \tag{18}$$

$$\boxed{\beta_S = \frac{-r_O r_S(\sin\theta_O\cos\theta_S - \sin\theta_S\cos\phi_S\cos\theta_O)}{r_O - r_S(\cos\theta_S\cos\theta_O + \sin\theta_O\sin\theta_S\cos\phi_S)}} \tag{19}$$

4 Magnification factors and positions of images.

The flux of an image of an infinitesimal source is the product of its surface brightness and the solid angle $\Delta\omega$ it subtends on the sky. Since the former quantity is unchanged during light deflection, the ratio of the flux of a sufficiently small image to that of its corresponding source in the absence of the lens, is given by

$$\mu = \frac{\Delta\omega}{(\Delta\omega)_0} = \frac{1}{|J|} \tag{20}$$

where 0-subscripts denote undeflected quantities and J is the Jacobian of the transformation $(x_{S,}\ y_S) \rightarrow (x_i, y_i)$ [1]. Writting $x_S = x_S(x_i, y_i), y_S = y_S(x_i, y_i)$ we can find expressions for the partial derivatives appearing in the Jacobian by differentiating equations (4) and (5). Indeed, the Jacobian is given by the expression:

$$J = xw - zy \tag{21}$$

where we defined: $\boxed{x := \frac{\partial x_S}{\partial x_i}, y := \frac{\partial x_S}{\partial y_i}, z := \frac{\partial y_S}{\partial x_i}, w := \frac{\partial y_S}{\partial y_i}}$. Writting equations (4) and (5) as follows(pvsol2) :

$$\begin{aligned} R_1(x_i, y_i) - A_1(x_i, y_i, x_S, y_S, m) &= 0 \\ \Delta\phi(x_S, y_S, n) - R_2(x_i, y_i) - A_2(x_i, y_i, x_S, y_S, m) &= 0 \end{aligned} \tag{22}$$

(wpf)we set up the following system of equations:

$$\beta_1 = -\alpha_1 x - \alpha_2 z \tag{23}$$

$$\beta_2 = -\alpha_1 y - \alpha_2 w \tag{24}$$

$$-\beta_3 = \alpha_3 x + \alpha_4 z \tag{25}$$

$$-\beta_4 = \alpha_3 y + \alpha_4 w \tag{26}$$

where $\boxed{\alpha_1 = \frac{\partial A_1}{\partial x_S},\ \alpha_2 = \frac{\partial A_1}{\partial y_S},\ \alpha_3 = -\frac{\partial \phi_s}{\partial x_S} - \frac{\partial A_2}{\partial x_S},\ \alpha_4 = -\frac{\partial \phi_s}{\partial y_S} - \frac{\partial A_2}{\partial y_S}}$,

$\boxed{\beta_1 = \frac{\partial R_1}{\partial x_i} - \frac{\partial A_1}{\partial x_i},\ \beta_2 = \frac{\partial R_1}{\partial y_i} - \frac{\partial A_1}{\partial y_i},\ \beta_3 = \frac{\partial R_2}{\partial x_i} + \frac{\partial A_2}{\partial x_i},\ \beta_4 = \frac{\partial R_2}{\partial y_i} + \frac{\partial A_2}{\partial y_i}}$.

Solving for x, y, z, w we obtain:

(Magnification)

$$\boxed{\mu = \frac{1}{|J|} = \left|\frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{\beta_1\beta_4 - \beta_2\beta_3}\right|} \tag{27}$$

5 Closed form solution for the angular integrals.

In this case we have to take into account the turning points in the polar coordinate. A generic angular polar integral can be written:

$$\pm\int_{\theta_1}^{\theta_2} = \int_{\min(z_1,z_2)}^{\max(z_1,z_2)} + [1 - \mathrm{sign}(\theta_1 \circ \theta_2)] \int_0^{\min(z_1,z_2)} \tag{28}$$

[1]Recall in the small angles approxiamation: $\alpha_i \approx r_O\ x_i$, $\beta_i \approx r_O\ y_i$. Also we define: $x_S := \frac{\alpha_S}{r_O}, y_S := \frac{\beta_S}{r_O}$.

where:

$$\theta_1 \circ \theta_2 := \cos\theta_1 \cos\theta_2 \tag{29}$$

Indeed, using the variable $z := \cos^2\theta$ we derive:

$$-\frac{1}{2}\frac{\mathrm{d}z}{\sqrt{z}}\frac{1}{\sqrt{1-z}} = \mathrm{sign}(\frac{\pi}{2}-\theta)\mathrm{d}\theta \tag{30}$$

This is the result of the fact that in the interval $0 \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$ and $\sin\theta \geq 0$, while in the interval $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin\theta \geq 0$, $\cos\theta \leq 0$. The angular integration in the polar variable includes the terms:

$$\int^{\theta} = \pm\int_{\theta_S}^{\theta_{\min\,/\,\max}} \pm \int_{\theta_{\min\,/\,\max}}^{\theta_{\max\,/\,\min}} \pm \int_{\theta_{\max\,/\,\min}}^{\theta_{\min\,/\,\max}} \pm\cdots\pm \int_{\theta_{\max\,/\,\min}}^{\theta_O}. \tag{31}$$

(mt)

The roots z_m, z_3 (of $\Theta(\theta)=0$) are expressed in terms of the integrals of motion and the cosmological constant by the expressions:

$$z_{m,3} = \frac{\mathcal{Q}+\Phi^2\Xi^2-H^2 \pm \sqrt{(\mathcal{Q}+\Phi^2\Xi^2-H^2)^2+4H^2\mathcal{Q}}}{-2H^2} \tag{32}$$

and(Lk)

$$H^2 := \frac{a^2\Lambda}{3}[\mathcal{Q}+(\Phi-a)^2\Xi^2]+a^2\Xi^2 \tag{33}$$

For $\Lambda = 0$, the turning points take the form:

$$\boxed{z_m = \frac{a^2-\mathcal{Q}-\Phi^2+\sqrt{4a^2\mathcal{Q}+(-a^2+\mathcal{Q}+\Phi^2)^2}}{2a^2},} \tag{34}$$

where the subscript "m" stands for "min/max". The corresponding angles are:

$$\boxed{\theta_{\mathrm{min/max}} = \mathrm{Arccos}(\pm\sqrt{z_m})} \tag{35}$$

.

Now for θ_j and $\theta_{\min\,/\,\max}$ in the same hemisphere:

$$\int_{\theta_j}^{\theta_{\min\,/\,\max}} \frac{\mathrm{d}\theta}{\pm\sqrt[2]{\Theta(\theta)}} = \frac{1}{2|a|}\int_{z_j}^{z_m}\frac{\mathrm{d}z}{\sqrt[2]{z(z_m-z)(z-z_3)}} \equiv I\_3 \tag{36}$$

Let us now calculate the elliptic integral in eqn.(36) in closed analytic form. Applying the transformation:

$$z = z_m + \xi^2(z_j - z_m) \tag{37}$$

our integral is calculated in closed form in terms of
Appell's generalized hypergeometric function F_1 of two variables:

$$I_3 = \frac{1}{2|a|}\frac{\sqrt[2]{(z_m - z_j)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m - z_j}{z_m},\frac{z_m - z_j}{z_m - z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \tag{38}$$

On the other hand using the transformation:

$$\boxed{z = \frac{u z_j z_m - z_j z_m}{u z_j - z_m}} \tag{39}$$

we calculate in closed form:

$$\begin{aligned}
&\frac{1}{2\,|a|}\int_0^{z_j}\frac{\mathrm{d}z}{\sqrt[2]{z(z_m - z)(z - z_3)}} \\
&= \frac{1}{|a|}\frac{\sqrt[2]{z_j}}{z_m}\sqrt[2]{\frac{z_j - z_m}{z_3 - z_j}} F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j}{z_m},\frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}\right) \\
&= \frac{1}{|a|}\frac{\sqrt[2]{\frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}}}{\sqrt[2]{z_m - z_3}} F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m - z_3}\frac{z_j(z_m - z_3)}{z_m(z_j - z_3)},\frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}\right)
\end{aligned} \tag{40}$$

In going from the second line to the third of (40) we made use of the following identity of Appell's first generalised hypergeometric function of two variables:

$$F_1(\alpha,\beta,\beta',\gamma,x,y) = (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}F_1(\gamma-\alpha,\beta,\gamma-\beta-\beta',\gamma,\frac{x-y}{x-1},y) \tag{41}$$

Likewise we derive the closed form solution for the following integral:

$$\begin{aligned}
&\frac{1}{2|a|}\int_0^{z_j}\frac{\mathrm{d}z}{(1-z)\sqrt[2]{z(z_m - z)(z - z_3)}} \\
&= \frac{z_j}{z_m}\frac{1}{|a|}\frac{z_j - z_m}{1 - z_j}\frac{1}{\sqrt[2]{z_j(z_j - z_m)(z_3 - z_j)}}\times \\
&\quad F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j(1 - z_m)}{z_m(1 - z_j)},\frac{z_j}{z_m},\frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}\right) \\
&= \frac{1}{|a|}\frac{z_j}{z_m}\sqrt[2]{\frac{z_m}{-z_3 z_j}}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},z_j,\frac{z_j}{z_m},\frac{z_j}{z_3}\right)
\end{aligned} \tag{42}$$

Producing the last line of equation (42) we used the following formula for the Lauricella function F_D (FD):

Proposition 1

$$F_D(\alpha,\beta,\beta',\beta'',\gamma,x,y,z)=(1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''}\times F_D\left(\gamma-\alpha,\beta,\gamma-\beta-\beta'-\beta'',\beta'',\gamma,\frac{x-y}{x-1},y,\frac{z-y}{z-1}\right)$$

Proof. Applying the transformation:

$$u=\frac{1-\nu}{1-\nu y} \tag{43}$$

onto the integral:

$$IR_{F_D}=\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-u\,y)^{-\beta'}(1-uz)^{-\beta''}\mathrm{d}u \tag{44}$$

we derive:

$$(1-u)^{\gamma-\alpha-1}=\left(\frac{\nu(1-y)}{1-\nu y}\right)^{\gamma-\alpha-1}, \quad (1-ux)^{-\beta}=\left(\frac{(1-x)[1-\frac{\nu(x-y)}{(x-1)}]}{1-\nu y}\right)^{-\beta}$$

$$(1-u\,y)^{-\beta'}=\frac{(1-y)^{-\beta'}}{(1-\nu y)^{-\beta'}}, \quad (1-uz)^{-\beta''}=\left(\frac{(1-z)[1-\frac{\nu(z-y)}{z-1}]}{1-\nu y}\right)^{-\beta''} \tag{45}$$

and thus we obtain the result:

$$IR_{F_D}=(1-y)^{\gamma-\alpha}(1-x)^{-\beta}(1-y)^{-\beta'}(1-z)^{-\beta''}\times \int_0^1 \mathrm{d}\nu\;\nu^{\gamma-\alpha-1}(1-\nu)^{\alpha-1}(1-\nu\,y)^{-(\gamma-\beta-\beta'-\beta'')}(1-\nu\frac{x-y}{x-1})^{-\beta}(1-\nu\frac{z-y}{z-1})^{-\beta''} \tag{46}$$

or

$$F_D(\alpha,\beta,\beta',\beta'',\gamma,x,y,z)=(1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''}\times F_D\left(\gamma-\alpha,\beta,\gamma-\beta-\beta'-\beta'',\beta'',\gamma,\frac{x-y}{x-1},y,\frac{z-y}{z-1}\right)$$

■

Likewise:

$$I_4:=\frac{-\Phi}{2|a|}\int_{z_j}^{z_m}\frac{\mathrm{d}z}{(1-z)\sqrt[2]{z(z_m-z)(z-z_3)}} = \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_j)}{z_m}}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j-z_m}{1-z_m},\frac{z_m-z_j}{z_m},\frac{z_m-z_j}{z_m-z_3}\right) \tag{47}$$

Let us see for instance the term in (31):

$$\pm\int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2\int_0^{z_m} \tag{48}$$

since $\cos^2\theta_{\min/\max} = z_m$ and $\theta_{\min}\circ\theta_{\max} = -z_m$.(angul.)

Equation (47) for $z_j = 0$, becomes (tpL):

$$\begin{aligned}
&-\frac{\Phi}{2|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{-z_m}{1-z_m},1,\frac{z_m}{z_m-z_3}\right)\\
&=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{1}{(1-z_m)}\frac{\pi}{2}F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{-z_m}{1-z_m},\frac{z_m}{z_m-z_3}\right)\\
&=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{\pi}{2}F_1\left(\frac{1}{2},1,-\frac{1}{2},1,\frac{z_m(1-z_3)}{z_m-z_3},\frac{z_m}{z_m-z_3}\right)\\
&=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{\pi}{2}\frac{1}{1-z_3}\left(F(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3})-z_3F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{z_m(1-z_3)}{z_m-z_3},\frac{z_m}{z_m-z_3}\right)\right)
\end{aligned} \tag{49}$$

On the other hand the angular integrals of the form $\pm\int_{\theta_S}^{\theta_{\min/\max}}$ in equation (5) are solved in closed analytic form as follows:

$$\begin{aligned}
\pm\int_{\theta_S}^{\theta_{\min/\max}} &= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\frac{z_S}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt[2]{z_S(z_S-z_m)(z_3-z_S)}}\times\\
&F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)
\end{aligned} \tag{50}$$

(isim)An equivalent expression for the above integral is(F):

$$
\begin{aligned}
\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}} &= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_S-z_m}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\sqrt[2]{\frac{z_S}{z_m}}\sqrt[2]{\frac{z_m-z_3}{z_S-z_3}}\frac{1}{\sqrt[2]{z_m-z_3}}\times\\
&F_D\left(\frac{1}{2},1,\frac{1}{2},-\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)\\
&= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_S-z_m}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\sqrt[2]{\frac{z_S}{z_m}}\sqrt[2]{\frac{z_m-z_3}{z_S-z_3}}\frac{1}{\sqrt[2]{z_m-z_3}}\times\\
&[\frac{-z_3}{1-z_3}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)+\\
&\frac{1}{1-z_3}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)]
\end{aligned}
\tag{51}
$$

Now, for a light trajectory that encounters m turning points ($m \geq 1$) we have:

$$
\boxed{\pm\int_{\theta_S}^{\theta_{\min/\max}}\underbrace{\pm\int_{\theta_{\min/\max}}^{\theta_{\max/\min}}\pm\int_{\theta_{\max/\min}}^{\theta_{\min/\max}}\cdots}_{m-1\text{ times}}\pm\int_{\theta_{\max/\min}}^{\theta_O}=}
\tag{52}
$$

$$
\begin{aligned}
&= \int_{z_S}^{z_m}+[1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\int_0^{z_S}\\
&+\int_{z_O}^{z_m}+[1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\int_0^{z_O}\\
&+2(m-1)\int_0^{z_m}
\end{aligned}
\tag{53}
$$

where:

$$
\boxed{\theta_{mO} := \mathrm{Arccos}(\mathrm{sign}(y_i)\sqrt{z_m}),}
\tag{54}
$$

and

$$
\theta_{mS} := \begin{cases}\theta_{mO}, & m\,\text{odd}\\ \pi-\theta_{mO}, & m\,\text{even}\end{cases}
\tag{55}
$$

Thus we have that (2len):

$$
\begin{aligned}
A_2(x_i,y_i,x_S,y_S,m) = 2(m-1)\times&\left[-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{1}{(1-z_m)}\frac{\pi}{2}F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{-z_m}{1-z_m},\frac{z_m}{z_m-z_3}\right)\right]\\
&+\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\times\\
&F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\frac{z_S}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt[2]{z_S(z_S-z_m)(z_3-z_S)}}\times\\
&F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_{3)}}\right)+\\
&+\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_O)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\times\\
&F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_O-z_m}{1-z_m},\frac{z_m-z_O}{z_m},\frac{z_m-z_O}{z_m-z_3}\right)\\
&[1-\mathrm{sign}(\theta_O\circ\theta_{mO})](-)\frac{\Phi}{|a|}\frac{z_O}{z_m}\frac{z_O-z_m}{1-z_O}\frac{1}{\sqrt[2]{z_O(z_O-z_m)(z_3-z_O)}}\times\\
&F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_O(1-z_m)}{z_m(1-z_O)},\frac{z_O}{z_m},\frac{z_O(z_m-z_3)}{z_m(z_O-z_{3)}}\right)
\end{aligned}
\tag{56}
$$

Lauricella's 4^{th} hypergeometric function of m-variables.

$$
\boxed{F_D(\alpha,\boldsymbol{\beta},\gamma,\mathbf{z}) = \sum_{n_1,n_2,\ldots,n_m=0}^{\infty}\frac{(\alpha)_{n_1+\cdots n_m}(\beta_1)_{n_1}\cdots(\beta_m)_{n_m}}{(\gamma)_{n_1+\cdots+n_m}(1)_{n_1}\cdots(1)_{n_m}}z_1^{n_1}\cdots z_m^{n_m}}
\tag{57}
$$

(tr1)

where

$$
\begin{aligned}
\mathbf{z} &= (z_1,\ldots,z_m),\\
\boldsymbol{\beta} &= (\beta_1,\ldots,\beta_m).
\end{aligned}
\tag{58}
$$

The Pochhammer symbol $\boxed{(\alpha)_m=(\alpha,m)}$ is defined by

$$
(\alpha)_m=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}=\begin{cases}1, & \text{if}\quad m=0\\ \alpha(\alpha+1)\cdots(\alpha+m-1) & \text{if}\, m=1,2,3\end{cases}
\tag{59}
$$

The series admits the following integral representation:

$$\boxed{F_D(\alpha,\boldsymbol{\beta},\gamma,\mathbf{z})=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z_1t)^{-\beta_1}\cdots(1-z_mt)^{-\beta_m}\mathrm{d}t} \tag{60}$$

which is valid for $\boxed{\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\gamma-\alpha)>0.}$. It converges absolutely inside the m-dimensional cuboid:

$$|z_j|<1,(j=1,\ldots,m). \tag{61}$$

The angular integrals of the form $\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}}$ in equation (4) are calculated in closed-analytic form as follows:

$$\begin{aligned}\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}}\frac{\mathrm{d}\theta}{\sqrt[2]{\Theta}}=&\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}\\&+[1-\mathrm{sign}(\theta_s\circ\theta_{ms})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times\\&F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\end{aligned} \tag{62}$$

Thus,

$$\begin{aligned}A_1(x_i,y_i,x_S,y_S,m)=&2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)+\\&\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}\\&+[1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times\\&F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+\\&\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_O)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_O}{z_m},\frac{z_m-z_O}{z_m-z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}\\&+[1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}}}{\sqrt[2]{z_m-z_3}}\times\\&F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)},\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}\right)\end{aligned} \tag{63}$$

6 Closed form solution for the radial integrals.

We now perform the radial integration assuming $\Lambda = 0$:(AO)

For an observer and a source located far away from the black hole, the relevant radial integrals can take the form:

$$\boxed{\int^{r} \rightarrow -\int_{r_S}^{\alpha} + \int_{\alpha}^{r_O} \simeq 2\int_{\alpha}^{\infty}} \tag{64}$$

For instance we meet the radial integral:

$$\int_{\alpha}^{\infty} \frac{aE}{\Delta}[(r^2+a^2) - a\Phi]\frac{dr}{\sqrt[2]{R}} \tag{65}$$

In order to calculate the contribution to the deflection angle from the radial term we need to integrate the above equation from the distance of closest approach (e.g., from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic by $\alpha, \beta, \gamma, \delta : \alpha > \beta > \gamma > \delta$. We manipulate first the terms:

$$\boxed{\int_{\alpha}^{\infty} \frac{a}{\Delta}\frac{(r^2+a^2)}{\sqrt{R}}\mathrm{d}r = \int_{\alpha}^{\infty} \frac{a\mathrm{d}r}{\sqrt{R}}\left[1 + \frac{\frac{2GM}{c^2}r}{\underbrace{r^2+a^2-\frac{2GM}{c^2}r}_{\Delta}}\right] = \int_{\alpha}^{\infty}\frac{a\mathrm{d}r}{\sqrt{R}} + \int_{\alpha}^{\infty}\frac{a\frac{2GMr}{c^2}\mathrm{d}r}{\Delta\sqrt{R}}} \tag{66}$$

Let us start with the term:

$$\int_{\alpha}^{\infty} \frac{a\frac{2GM}{c^2}r - a^2\Phi}{\Delta\sqrt[2]{R}}\mathrm{d}r \tag{67}$$

Expressing the roots of Δ as r_+, r_-, which are the radii of the event horizon and the inner or Cauchy horizon, and using partial fractions we derive the expression:

$$\int_\alpha^\infty \frac{a\frac{2GM}{c^2}r-a^2\Phi}{\Delta\sqrt[2]{R}}\mathrm{d}r=\int_\alpha^\infty \frac{A_+^{go}}{(r-r_+)\sqrt[2]{R}}\mathrm{d}r+\int_\alpha^\infty \frac{A_-^{go}}{(r-r_-)\sqrt[2]{R}}\mathrm{d}r$$
$$=\int_\alpha^\infty \frac{A_+^{go}}{(r-r_+)\sqrt[2]{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}}\mathrm{d}r$$
$$+\int_\alpha^\infty \frac{A_-^{go}}{(r-r_-)\sqrt[2]{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}}\mathrm{d}r \tag{68}$$

where $A_\pm^{go}$ are given by the equations

$$A_\pm^{go}=\pm\frac{(r_\pm a2\frac{GM}{c^2}-a^2\Phi)}{r_+-r_-} \tag{69}$$

For polar orbits $\Phi=0$ and the coefficients in (69) reduce to those calculated in Kraniotis,CQG22(2005)4391–4424.

We organize all roots in ascending order of magnitude as follows[2],

$$\alpha_\mu>\alpha_\nu>\alpha_i>\alpha_\rho \tag{70}$$

where $\alpha_\mu=\alpha_{\mu+1},\alpha_\nu=\alpha_{\mu+2},\alpha_\rho=\alpha_\mu$ and $\alpha_i=\alpha_{\mu-i},i=1,2,3$ and we have that $\alpha_{\mu-1}\geq\alpha_{\mu-2}>\alpha_{\mu-3}$. By applying the transformation

$$r=\frac{\omega z\alpha_{\mu+2}-\alpha_{\mu+1}}{\omega z-1} \tag{71}$$

or equivalently

$$z=\left(\frac{\alpha_\mu-\alpha_{\mu+2}}{\alpha_\mu-\alpha_{\mu+1}}\right)\left(\frac{r-\alpha_{\mu+1}}{r-\alpha_{\mu+2}}\right) \tag{72}$$

where

$$\omega:=\frac{\alpha_\mu-\alpha_{\mu+1}}{\alpha_\mu-\alpha_{\mu+2}} \tag{73}$$

we can bring our radial integrals into the familiar integral representation of Lauricella's F_D and Appell's hypergeometric function F_1 of three and two variables respectively. Indeed, we derive

[2]We have the correspondence $\alpha_{\mu+1}=\alpha,\alpha_{\mu+2}=\beta,\alpha_{\mu-1}=r_+=\alpha_{\mu-2},\alpha_{\mu-3}=\gamma,\alpha_\mu=\delta$.

$$\Delta\phi_{r_1}^{go} = 2\Bigg[\int_0^{1/\omega}\frac{-A_+^{go}\omega(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^+}\frac{\mathrm{d}z}{\sqrt[2]{z(1-z)}(1-\kappa_+^2 z)\sqrt[2]{1-\mu^2 z}}$$
$$+\int_0^{1/\omega}\frac{A_+^{go}\omega^2(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^+}\frac{z\mathrm{d}z}{\sqrt[2]{z(1-z)}(1-\kappa_+^2 z)\sqrt[2]{1-\mu^2 z}}$$
$$+\int_0^{1/\omega}\frac{-A_-^{go}\omega(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^-}\frac{\mathrm{d}z}{\sqrt[2]{z(1-z)}(1-\kappa_-^2 z)\sqrt[2]{1-\mu^2 z}}$$
$$+\int_0^{1/\omega}\frac{A_-^{go}\omega^2(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^-}\frac{z\mathrm{d}z}{\sqrt[2]{z(1-z)}(1-\kappa_-^2 z)\sqrt[2]{1-\mu^2 z}}\Bigg] \quad (74)$$

where the moduli $\kappa_\pm^2, \mu^2$ are

$$\kappa_\pm^2 = \left(\frac{\alpha_\mu-\alpha_{\mu+1}}{\alpha_\mu-\alpha_{\mu+2}}\right)\left(\frac{\alpha_{\mu+2}-\alpha_{\mu-1}^\pm}{\alpha_{\mu+1}-\alpha_{\mu-1}^\pm}\right), \qquad \mu^2 = \left(\frac{\alpha_\mu-\alpha_{\mu+1}}{\alpha_\mu-\alpha_{\mu+2}}\right)\left(\frac{\alpha_{\mu+2}-\alpha_{\mu-3}}{\alpha_{\mu+1}-\alpha_{\mu-3}}\right) \quad (75)$$

Also

$$H^\pm = \sqrt[2]{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})(\alpha_{\mu+1}-\alpha_{\mu-1}^\pm)\sqrt[2]{\alpha_{\mu+1}-\alpha_\mu}\sqrt[2]{\alpha_{\mu+1}-\alpha_{\mu-3}} \quad (76)$$

and $\alpha_{\mu-1}^\pm = r_\pm$. By defining a new variable $z' := \omega z$ we can express the contribution $\Delta\phi_{r_1}^{go}$,to the deflection angle, from the above radial terms in terms of Lauricella's hypergeometric function F_D

$$\Delta\phi_{r_1}^{go} = 2\Bigg[\frac{-2A_+^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^+}F_D\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{3}{2},\frac{1}{\omega},\kappa_+'^2,\mu'^2\right)$$
$$+\frac{A_+^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^+}F_D\left(\frac{3}{2},\frac{1}{2},1,\frac{1}{2},\frac{5}{2},\frac{1}{\omega},\kappa_+'^2,\mu'^2\right)\frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)}$$
$$+\frac{-2A_-^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^-}F_D\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{3}{2},\frac{1}{\omega},\kappa_-'^2,\mu'^2\right)$$
$$+\frac{A_-^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^-}F_D\left(\frac{3}{2},\frac{1}{2},1,\frac{1}{2},\frac{5}{2},\frac{1}{\omega},\kappa_-'^2,\mu'^2\right)\frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)}\Bigg] \quad (77)$$

where the variables of the function F_D are given in terms of the roots of the quartic and the radii of the event and Cauchy horizons by the expressions

$$\begin{aligned}
\frac{1}{\omega} &= \frac{\alpha_\mu - \alpha_{\mu+2}}{\alpha_\mu - \alpha_{\mu+1}} = \frac{\delta - \beta}{\delta - \alpha} \\
\kappa_\pm'^2 &= \frac{\alpha_{\mu+2} - \alpha_{\mu-1}^\pm}{\alpha_{\mu+1} - \alpha_{\mu-1}^\pm} = \frac{\beta - r_\pm}{\alpha - r_\pm} \\
\mu'^2 &= \frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}} = \frac{\beta - \gamma}{\alpha - \gamma}
\end{aligned} \tag{78}$$

An equivalent expression is as follows

$$\begin{aligned}
\Delta\phi_{r_1}^{go} = 2\Bigg[& \frac{-2A_+^{go}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_+'^2, \mu'^2\right) \\
& + \frac{A_+^{go}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+}\Bigg(-\frac{1}{\kappa_+'^2} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu'^2\right) 2 \\
& + \frac{1}{\kappa_+'^2} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_+'^2, \mu'^2\right) 2 \Bigg) \\
& + \frac{-2A_-^{go}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_-'^2, \mu'^2\right) \\
& + \frac{A_-^{go}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-}\Bigg(-\frac{1}{\kappa_-'^2} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu'^2\right) 2 \\
& + \frac{1}{\kappa_-'^2} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_-'^2, \mu'^2\right) 2 \Bigg) \Bigg]
\end{aligned} \tag{79}$$

In going from (77) to (79) we used the identity proven in:
Kraniotis,CQG22(2005)4391-4424

$$\begin{aligned}
F_D\left(\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2}, \frac{1}{\omega}, \kappa_+'^2, \mu'^2\right)\frac{\Gamma(3/2)}{\Gamma(5/2)} = & -\frac{1}{\kappa_+'^2} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu'^2\right)\frac{\Gamma(1/2)}{\Gamma(3/2)} \\
& + \frac{1}{\kappa_+'^2} F_D\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_+'^2, \mu'^2\right)\frac{\Gamma(1/2)}{\Gamma(3/2)}
\end{aligned} \tag{80}$$

Finally the term $a\int_\alpha^\infty \frac{dr}{\sqrt{R}}$ gives the following contribution :$\Delta\phi_{r_2}^{go}$

$$\Delta\phi^{go}_{r_2} = a\frac{(\alpha_{\mu+1}-\alpha_{\mu+2})}{\sqrt[2]{(\alpha_{\mu+2}-\alpha_{\mu+1})^2(\alpha_{\mu-1}-\alpha_{\mu+1})(\alpha_\mu-\alpha_{\mu+1})}} \times$$
$$\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\lambda_1^2,\frac{1}{\omega}\right) \tag{81}$$

We exploit further the lens-equations (22). For θ_{mS},θ_S on the same hemi-sphere:

$$R_1(x_i,y_i)-2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)+\cdots=\int^{\xi_S}\frac{\mathrm{d}\xi}{\sqrt{4\xi^3-g_2\xi-g_3}} \tag{82}$$

Inverting:

$$\xi_S=\wp\left((81)/a-2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)+\cdots+\epsilon\right) \tag{83}$$

and then:

$$-\phi_S=R_2(x_i,y_i)+A_2(x_i,y_i,x_S,y_S,m) \tag{84}$$

where $\wp(z)$ denotes the Weierstraß elliptic function (which is also a meromoprhic Jacobi modular form of weight 2) and the Weierstraß invariants are given by:

$$g_2=\frac{1}{12}(\alpha+\beta)^2-\mathcal{Q}\frac{\alpha}{4}, \tag{85}$$
$$g_3=\frac{1}{216}(\alpha+\beta)^3-\mathcal{Q}\frac{\alpha^2}{48}-\mathcal{Q}\frac{\alpha\beta}{48} \tag{86}$$

while $\alpha:=-a^2,\beta:=\mathcal{Q}+\Phi^2, z_S=-\frac{\xi_S+\frac{\alpha+\beta}{12}}{-\alpha/4}$. Also ϵ denotes a constant of integration.

7 Magnifications for an equatorial observer in a Kerr black hole.

In this case $(\theta_O=\pi/2)$, equations (15),(16), become:

$$\Phi\simeq-\alpha_i\sin\theta_O=-\alpha_i \tag{87}$$
$$\mathcal{Q}\simeq\beta_i^2+(\alpha_i^2-a^2)\cos^2\theta_O=\beta_i^2 \tag{88}$$

and

$$x_S:=\frac{\alpha_S}{r_O}=\frac{r_S\sin\theta_S\sin\phi_S}{r_O-r_S\sin\theta_S\cos\phi_S} \tag{89}$$

$$\boxed{y_S := \frac{\beta_S}{r_O} = \frac{-r_S\cos\theta_S}{r_O - r_S\sin\theta_S\cos\phi_S}} \tag{90}$$

$$\begin{aligned}
\frac{\partial(51)}{\partial x_S} &= \frac{\partial(51)}{\partial z_S}\frac{\partial z_S}{\partial x_S},\\
\frac{\partial(51)}{\partial z_S} &= \frac{\Phi}{2|a|}\frac{1}{z_m}\frac{1}{(1-z_m)}\frac{1}{\sqrt{z_m-z_3}}\left(\frac{z_m-z_S}{z_m}\right)^{-1/2}\times\\
&F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+\\
&\left(\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\right)\times\Bigg\{\\
&F_D\left(\frac{3}{2},2,\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{1}{1-z_m}+\\
&F_D\left(\frac{3}{2},1,\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{-1}{z_m}+\\
&F_D\left(\frac{3}{2},1,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{-1}{z_m-z_3}\Bigg\}+\\
&(1-\mathrm{sign}(\theta_S\circ\theta_{ms}))(-1)\Bigg[\Big[\frac{1}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}+\\
&\frac{z_S}{z_m}\frac{z_3(z_m-3z_Sz_m+2z_S^2)-z_S(z_m(2-4z_S)+z_S(-1+3z_S))}{2(1-z_S)^2z_S(z_3-z_S)\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\Big]\times\\
&F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+\\
&\frac{z_S}{z_m}\frac{z_S-z_m}{(1-z_S)}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\Bigg\{\\
&F_D\left(2,2,-\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\frac{1-z_m}{z_m(1-z_S)^2}+\\
&F_D\left(2,1,\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\frac{1}{z_m}+\\
&F_D\left(2,1,\frac{-1}{2},\frac{3}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{-z_3(z_m-z_3)}{z_m(z_S-z_3)^2}\right)\Bigg\}\Bigg]
\end{aligned} \tag{91}$$

Now we calculate the term: $\frac{\partial(62)}{\partial z_S}$. Indeed, calculating the derivatives w.r.t. z_S we derive the expression:

$$\begin{aligned}\frac{\partial(62)}{\partial z_S} = & \frac{1}{2|a|}\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)}\left(-\frac{1}{2\sqrt{z_m(z_m-z_3)}\sqrt{z_m-z_S}}\right)F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+\\ & \frac{1}{2|a|}\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)}\sqrt{\frac{(z_m-z_S)}{z_m(z_m-z_3)}}\times\left[F_1\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\left(\frac{-1}{z_m}\right)+\right.\\ & \left.F_1\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\left(\frac{-1}{z_m-z_3}\right)\right]+\\ & [1-\mathrm{sign}(\theta_S\circ\theta_{ms})]\left[\frac{1}{2|a|}\left(\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)^{-\frac{1}{2}}\left\{\frac{(-z_3)\sqrt{z_m-z_3}}{z_m(z_S-z_3)^2}\right\}\times\right.\\ & F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+\\ & \frac{1}{|a|}\frac{\sqrt{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt{z_m-z_3}}\times\left[F_1\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{-z_3}{(z_S-z_3)^2}\right)+\right.\\ & \left.\left.F_1\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{(-z_3)(z_m-z_3)}{z_m(z_S-z_3)^2}\right)\right]\right]\end{aligned} \tag{92}$$

$$\boxed{\alpha_1=\frac{\partial A_1}{\partial x_S}=(92)\times\frac{\partial z_S}{\partial x_S}=(92)\times\left(-2\cos\theta_S\sin\theta_S\times\frac{\frac{r_S^2\sin\theta_S\cos\theta_S\sin\phi_S}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right)} \tag{93}$$

$$\boxed{\alpha_2=\frac{\partial A_1}{\partial y_S}=(92)\times\frac{\partial z_S}{\partial y_S}=(92)\times\left(-2\cos\theta_S\sin\theta_S\times\frac{\frac{-[r_Or_S\sin\theta_S\cos\phi_S-r_S^2\sin^2\theta_S]}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right)} \tag{94}$$

While for the α_3,α_4 terms which contrbute to the expression for the magnification, equation (27), m we derive the expressions:

$$\boxed{\alpha_3=-\frac{\partial\phi_S}{\partial x_S}-\frac{\partial A_2}{\partial x_S}=-\left(-\frac{\frac{(r_Or_S\sin\theta_S-r_S^2\cos\phi_S)}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right)-(91)\times\left(\frac{\frac{r_S^2\sin\theta_S\cos\theta_S\sin\phi_S}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right)} \tag{95}$$

,

$$\alpha_4 = -\frac{\partial \phi_S}{\partial y_S} - \frac{\partial A_2}{\partial y_S} = -\frac{\frac{r_O r_S \cos\theta_S \sin\phi_S}{(r_O - r_S \sin\theta_S \cos\phi_S)^2}}{J_1} - (91) \times \frac{\frac{-[r_O r_S \sin\theta_S \cos\phi_S - r_S^2 \sin^2\theta_S]}{(r_O - r_S \sin\theta_S \cos\phi_S)^2}}{J_1} \Bigg) \tag{96}$$

where J_1 denotes the Jacobian:

$$J_1 = \frac{\partial(x_S, y_S)}{\partial(\theta_S, \phi_S)} \tag{97}$$

and

$$\begin{aligned}
\frac{\partial \theta_S}{\partial x_S} &= \frac{(r_S^2 \sin\theta_S \cos\theta_S \sin\phi_S)/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial \theta_S}{\partial y_S} &= \frac{-[r_O r_S \sin\theta_S \cos\phi_S - r_S^2 \sin^2\theta_S]/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial \phi_S}{\partial x_S} &= -\frac{(r_O r_S \sin\theta_S - r_S^2 \cos\phi_S)/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial \phi_S}{\partial y_S} &= \frac{r_O r_S \cos\theta_S \sin\phi_S/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1}
\end{aligned} \tag{98}$$

In producing the results exhibited in eqns (91),(92) in our calculations for the magnification factors we made use of the important identity of Appell's hypergeometric function F_1 and its corresponding generalization for the Lauricella hypergeometric function F_D :

$$\frac{\partial^{m+n} F_1(\alpha,\beta,\beta',\gamma,x,y)}{\partial x^m \partial x^n} = \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)} \times F_1(\alpha+m+n,\beta+m,\beta'+n,\gamma+m+n,x,y) \tag{99}$$

Similar calculations that we do not exhibit in this talk leads to the derivation of the coefficients β_i.

8 Exact solution of the angular integrals in the presence of Λ.

There has been a discussion in the literature as to whether or not the cosmological constant contributes to the gravitational lensing. However, the debate has been restricted to the Schwarzschild-de Sitter spacetime Lake (2007), Sereno,Phys.Rev.D 77(2008), Rindler,Phys.Rev.D76(2007). Let us discuss now the more general case of gravitational lensing in the Kerr-de Sitter spacetime.

The generalized solution for the angular integral (50) in the presence of Λ is given by:

$$\pm\int_{\theta_S}^{\theta_{\min/\max}} = \frac{\Xi^2\Phi}{2|H|}\frac{z_S-z_m}{(1-\eta z_m)(1-z_m)}\frac{1}{\sqrt[2]{z_m(z_m-z_S)(z_m-z_3)}}\times$$
$$F_D\left(\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{\eta(z_S-z_m)}{1-\eta z_m},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+$$
$$(1-\mathrm{sign}(\theta_S\circ\theta_{mS}))\left[-\frac{\Xi^2}{|H|}\frac{z_S}{z_m}\frac{z_S-z_m}{\eta z_S-1}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\times\right.$$
$$\left.\left\{aF_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\lambda,\frac{z_S}{z_m},\mu\right)+\frac{\Phi}{z_j-1}F_D(1,1,1,-\frac{3}{2},\frac{1}{2},\frac{3}{2},\lambda,\nu,\frac{z_S}{z_m},\mu\right\}\right] \tag{100}$$

where (H):

$$\eta:=-\frac{a^2\Lambda}{3},\mu=\frac{z_S}{z_m}\frac{z_m-z_3}{z_S-z_3},\lambda=\frac{z_S}{z_m}\left(\frac{1-\eta z_m}{1-\eta z_S}\right),\nu=\frac{z_S}{z_m}\left(\frac{1-z_m}{1-z_S}\right)$$

Also the integrals $\pm\int_{\theta_{\min/\max}}^{\theta_{\max/\min}}=2\int_0^{z_m}$ contribute the term:

$$2(m-1)\times\frac{\Xi^2\Phi}{2|H|}\frac{-z_m}{(1-\eta z_m)(1-z_m)}\frac{1}{\sqrt{z_m^2(z_m-z_3)}}$$
$$\times F_D\left(\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{\eta(-z_m)}{1-\eta z_m},\frac{-z_m}{1-z_m},1,\frac{z_m}{z_m-z_3}\right) \tag{101}$$

Notice that for $\Lambda=0$ this reduces to equation (49).

9 Closed-form solution for radial integrals in the presence of Λ.

Assume first $\Lambda>0$. We need to calculate radial integrals of the form:

$$\int\frac{a\Xi^2}{\Delta_r}((r^2+a^2)-a\Phi))\frac{\mathrm{d}r}{\sqrt[2]{R}} \tag{102}$$

We use the technique of partial fractions from integral calculus:

$$\frac{a\Xi^2}{\Delta_r}((r^2+a^2)-a\Phi))=\frac{A^1}{r-r_\Lambda^+}+\frac{A^2}{r-r_\Lambda^-}+\frac{A^3}{r-r_+}+\frac{A^4}{r-r_-} \tag{103}$$

where $r_\Lambda^+,r_\Lambda^-,r_+,r_-$ are the four real roots of Δ_r (Der).

For instance, for $r_O,r_S<r_\Lambda^+$ one of the integrals we need to calculate is:

$$\frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)}}\int_\alpha^{r_\Lambda^+/2}\frac{A^1\mathrm{d}r}{(r-r_\Lambda^+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} \tag{104}$$

and

$$\int_{\alpha}^{r_\Lambda^+/2} \frac{A^1 \mathrm{d}r}{(r-r_\Lambda^+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}}$$
$$= \frac{\rho_1}{\sqrt{\rho_1}} H_\Lambda^+ \times$$
$$F_D\left(\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{\beta-\gamma}{\alpha-\gamma}\frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{\beta-\delta}{\alpha-\delta}\frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{r_\Lambda^+ - \beta}{r_\Lambda^+ - \alpha}\frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}\right)\frac{\Gamma(1/2)}{\Gamma(3/2)} \tag{105}$$

where

$$\rho_1 := \frac{r_\Lambda^+ - \beta}{r_\Lambda^+ - \alpha}\frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta} \tag{106}$$

Similar calculations were performed for the rest of the radial integrals.

10 Conclusions.

- The precise analytic treatment of Kerr and Kerr-de Sitter black holes as gravitational lenses has been achieved. Full analytic strong-field calculation for the magnification factors was performed.
- Λ does contribute to the gravitational bending of light.
- Important application to the Sgr A* galactic centre black hole (Ghez).
- Fruitfull synergy of various fields of Science: general relativity, astrophysics, cosmology, pure mathematics(Pi1),(Pi2).