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Scalar-tensor cosmologies with a potential in the general relativity limit

Laur Järv[1]

Piret Kuusk,[1] Margus Saal,[2]
[1] University of Tartu, [2] Tartu Observatory, Estonia

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Outline

General goal: explaining dark energy, testing gravity at large scales.
Notice:

- Many proposed modified gravity theories can be cast in the form of scalar-tensor gravity (STG) - higher dimensions, branes, $f(R)$, VSL
- "Attractor mechanism" - wide classes of STG cosmologies dynamically converge to fixed points (Damour, Nordtvedt 1993).

Present work:

- Determine the conditions for attractive fixed points in STG cosmology
- Find the general analytic form of solutions around these fixed points

Therefore can:

- Use these to confront observations (local weak field, expansion history, growth of perturbations)
- Have a selection principle

Scalar-tensor gravity

Our starting point: "Brans-Dicke" parametrization, Jordan frame

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\Psi R[g_{\mu\nu}] - \frac{\omega(\Psi)}{\Psi}\nabla^\rho\Psi\nabla_\rho\Psi - 2\kappa^2 V(\Psi)\right] + S_m \quad (1)$$

- Family of theories, each pair $\omega(\Psi)$ and $V(\Psi)$ specifies a theory
- Variable gravitational "constant" $8\pi G = \frac{\kappa^2}{\Psi}$, assume $0 < \Psi < \infty$
- Assume positive energy denisty: $2\omega(\Psi) + 3 \geq 0$, $V(\Psi) \geq 0$
- S_m usual matter

Local weak field experiments (Solar System)

If $V(\Psi)$ can be neglected, then

$$8\pi G = \frac{\kappa^2}{\Psi}\frac{2\omega+4}{2\omega+3} \tag{2}$$

$$\beta - 1 = \frac{\kappa^2}{G}\frac{\frac{d\omega}{d\Psi}}{(2\omega+3)^2(2\omega+4)} \lesssim 10^{-4} \tag{3}$$

$$\gamma - 1 = -\frac{1}{\omega+2} \lesssim 10^{-5} \tag{4}$$

$$\frac{\dot{G}}{G} = -\dot{\Psi}\frac{2\omega+3}{2\omega+4}\left(G + \frac{2\frac{d\omega}{d\Psi}}{(2\omega+3)^2}\right) \lesssim 10^{-13}\ \mathrm{yr}^{-1} \tag{5}$$

"The limit of general relativity", "Nordtvedt limit" (1970)

$$\frac{1}{2\omega+3} \to 0\,, \qquad \frac{\frac{d\omega}{d\Psi}}{(2\omega+3)^3} \to 0\,. \tag{6}$$

(If $V(\Psi)$ gives a contribution, then the PPN parameters get a correction Olmo 2005, Perivolaropoulos 2009.)

Scalar-tensor cosmology

Flat ($k = 0$) FLRW, barotropic matter fluid $p = \mathrm{w}\rho$

$$H^2 = -H\frac{\dot\Psi}{\Psi} + \frac{1}{6}\frac{\dot\Psi^2}{\Psi^2}\,\omega(\Psi) + \frac{\kappa^2}{3}\frac{\rho}{\Psi} + \frac{\kappa^2}{3}\frac{V(\Psi)}{\Psi}\,, \tag{7}$$

$$2\dot H + 3H^2 = -2H\frac{\dot\Psi}{\Psi} - \frac{1}{2}\frac{\dot\Psi^2}{\Psi^2}\,\omega(\Psi) - \frac{\ddot\Psi}{\Psi} - \frac{\kappa^2}{\Psi}\mathrm{w}\rho + \frac{\kappa^2}{\Psi}\;V(\Psi)\,, \tag{8}$$

$$\ddot\Psi = -3H\dot\Psi - \frac{1}{2\omega(\Psi)+3}\;\frac{d\omega(\Psi)}{d\Psi}\;\dot\Psi^2 + \frac{\kappa^2}{2\omega(\Psi)+3}\;(1-3\mathrm{w})\;\rho + \frac{2\kappa^2}{2\omega(\Psi)+3}\;\left[2V(\Psi) - \Psi\;\frac{dV(\Psi)}{d\Psi}\right]\,, \tag{9}$$

$$\dot\rho = -3H\;(\mathrm{w}+1)\;\rho \tag{10}$$

STG cosmology as a dynamical system

Phase space: $\{\Psi, \Pi = \dot{\Psi}, H, \rho\}$, tangent of trajectories: $(\dot{\Psi}, \dot{\Pi}, \dot{H}, \dot{\rho})$

$$\dot{\Psi} = \Pi \,, \tag{11}$$

$$\dot{\Pi} = -\frac{1}{2\omega(\Psi)+3}\left[\frac{d\omega(\Psi)}{d\Psi}\Pi^2 - \kappa^2\,(1-3\mathrm{w})\,\rho + 2\kappa^2\left(\frac{dV(\Psi)}{d\Psi}\Psi - 2V(\Psi)\right)\right] - 3H\Pi \,, \tag{12}$$

$$\dot{H} = \frac{1}{2\Psi(2\omega(\Psi)+3)}\left[\frac{d\omega(\Psi)}{d\Psi}\Pi^2 - \kappa^2\,(1-3\mathrm{w})\,\rho + 2\kappa^2\left(\frac{dV(\Psi)}{d\Psi}\Psi - 2V(\Psi)\right)\right] - \frac{1}{2\Psi}\left[6H^2\Psi + 2H\Pi - \kappa^2(1-\mathrm{w})\rho - 2\kappa^2 V(\Psi)\right] \,, \tag{13}$$

$$\dot{\rho} = -3H(1+\mathrm{w})\rho \,. \tag{14}$$

STG cosmology phase space

The $k = 0$ trajectories lie on the 3-surface

$$H = -\frac{\Pi}{2\Psi} \pm \sqrt{(2\omega(\Psi)+3)\,\frac{\Pi^2}{12\Psi^2} + \frac{\kappa^2(\rho + V(\Psi))}{3\Psi}}\,, \qquad (15)$$

Boundaries in the phase space:

- $|H| \to \infty$, $|\rho| \to \infty$, or $|\dot{\Psi}| \to \infty$ imply a spacetime curvature singularity,
- $\Psi \to 0$ generally also a singularity (can not slip from "attractive" to "repulsive" gravity),
- $\Psi \to \infty$ not a singularity, but gravitational "constant" $\frac{\kappa^2}{\Psi}$ vanishes,
- $V \to \infty$ or $2\omega + 3 \to 0$ again a singularity,
- $\frac{1}{2\omega+3} \to 0$ turns out to be a singularity as well, unless $\dot{\Psi} = \Pi \to 0$.

JKS 2008

If potential dominates over matter density ($V \not\equiv 0$, $\rho \equiv 0$)

Using (15) can eliminate H and obtain a 2-dimensional system:

$$\dot{\Psi} = \Pi \tag{16}$$

$$\dot{\Pi} = \left(\frac{3}{2\Psi} - \frac{1}{2\omega(\Psi)+3}\,\frac{d\omega}{d\Psi}\right)\Pi^2 + \frac{2\kappa^2}{2\omega(\Psi)+3}\left(2V(\Psi) - \Psi\,\frac{dV}{d\Psi}\right) \mp \frac{1}{2\Psi}\sqrt{3(2\omega(\Psi)+3)\Pi^2 + 12\kappa^2\Psi V(\Psi)}\;\Pi\,. \tag{17}$$

Study the behavior of trajectories.

Fixed points ($V \not\equiv 0$, $\rho \equiv 0$ case)

Fixed points ($\dot\Psi = 0, \dot\Pi = 0$) are of two types, given by:

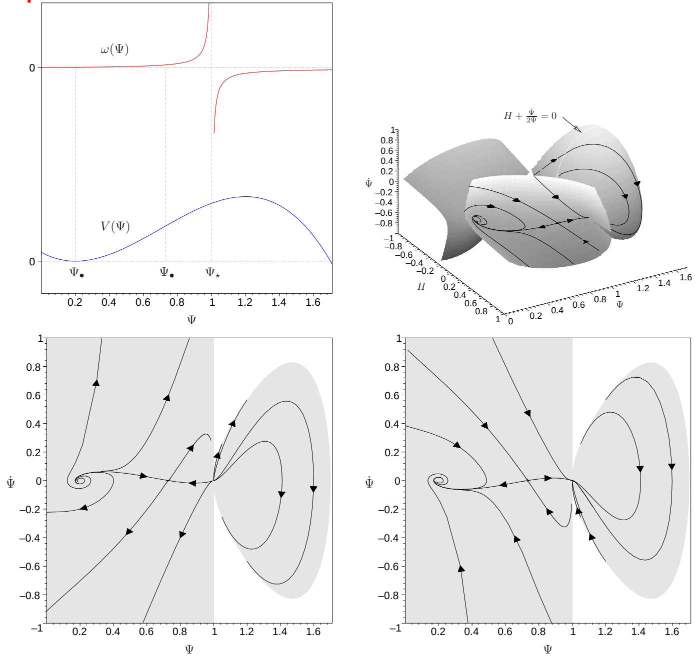
$$\Psi_\bullet \quad : \quad \frac{dV}{d\Psi}\Big|_{\Psi_\bullet} \Psi_\bullet - 2V(\Psi_\bullet) = 0\,, \tag{18}$$

$$\Psi_\star \quad : \quad \frac{1}{2\omega(\Psi_\star)+3} = 0\,, \quad \frac{1}{(2\omega(\Psi_\star)+3)^2}\frac{d\omega}{d\Psi}\Big|_{\Psi=\Psi_\star} \neq 0, \tag{19}$$

The properties of fixed points (node, focus, saddle; stable, unstable) and the form of solutions around the fixed points are determined by the eigenvalues, and these by $\omega(\Psi_{\bullet,\star})$, $\frac{d\omega}{d\Psi}|_{\Psi_{\bullet,\star}}$, $V(\Psi_{\bullet,\star})$, $\frac{dV}{d\Psi}|_{\Psi_{\bullet,\star}}$, $\frac{d^2V}{d\Psi^2}|_{\Psi_{\bullet,\star}}$ JKS 2008.

Notice $\Psi_\star$ is compatible with the "Nordtvedt limit", i.e. the local weak field experments.

Example



$\omega(\Psi)$
$V(\Psi)$
$\Psi_\bullet$
$\Psi_\bullet$
$\Psi_\star$
Ψ
$\dot\Psi$
H
$H + \frac{\dot\Psi}{2\Psi} = 0$

If matter density dominates over potential ($\rho \not\equiv 0$, $V \equiv 0$)

Use new time variable $dp \equiv \left| H + \frac{\dot{\Psi}}{2\Psi} \right| dt$, can eliminate H, to get

$$\Psi' = \Upsilon \quad (20)$$

$$\Upsilon' = \pm \frac{2\omega(\Psi) + 3}{8\Psi^2} \Upsilon^3 + \frac{6\omega(\Psi) + 9 - 4\Psi \frac{d\omega(\Psi)}{d\Psi}}{4\Psi(2\omega(\Psi) + 3)} \Upsilon^2 \mp \frac{3}{2} \Upsilon + \frac{3\Psi}{2\omega(\Psi) + 3} \quad (21)$$

Fixed point ($\Psi' = 0, \Upsilon' = 0$) in p-time corresponds to a fixed point ($\dot{\Psi} = 0, \dot{\Pi} = 0$) in t-time, and is given by:

$$\Psi_\star \quad : \quad \frac{1}{2\omega(\Psi_\star) + 3} = 0 \,, \quad \frac{1}{(2\omega(\Psi_\star) + 3)^2} \frac{d\omega}{d\Psi}\Big|_{\Psi = \Psi_\star} \neq 0, \quad (22)$$

Again it is compatible with the "Nordtvedt limit".

A more careful analysis of $\Psi_\star$

In the limit $(\Psi_\star, \Pi_\star)$: (a) $\frac{1}{2\omega(\Psi)+3} \to 0$, (b) $\dot{\Psi} \equiv \Pi \to 0$ the equations contain an indeterminacy (like $\frac{y}{x}$ at the origin).

Let us focus around this point $\Psi = \Psi_\star + x$, $\Pi = \Pi_\star + y = y$ and expand in series

$$\frac{1}{2\omega(\Psi)+3} = \frac{1}{2\omega(\Psi_\star)+3} + A_\star x + \ldots \approx A_\star x \,, \tag{23}$$

$$(2\omega(\Psi)+3)\Pi^2 = \frac{y^2}{0 + A_\star x + \ldots} = \frac{y^2}{A_\star x}(1 + O(x)) \approx \frac{y^2}{A_\star x} \,, \tag{24}$$

where (c) $A_\star \equiv \frac{d}{d\Psi}\left(\frac{1}{2\omega(\Psi)+3}\right)\Big|_{\Psi_\star} \neq 0$ ja (d) $\frac{1}{2\omega(\Psi)+3}$ is differentiable at $\Psi_\star$.

Non-linear approximation ($V \neq 0$, $\rho \equiv 0$ case)

Keeping terms which are of first order in x and y, the dynamical system (16), (17) becomes

$$\dot{x} = y\,, \tag{25}$$

$$\dot{y} = \frac{y^2}{2x} - C_1\, y + C_2\, x\,, \tag{26}$$

where

$$C_1 \equiv \pm\sqrt{\frac{3\kappa^2 V(\Psi_\star)}{\Psi_\star}}\,, \qquad C_2 \equiv 2\kappa^2 A_\star \left(2V(\Psi) - \frac{dV(\Psi)}{d\Psi}\Psi\right)\Big|_{\Psi_\star}\,, \tag{27}$$

encode the behavior of the functions ω and V near this point.
JKS 2010a

Phase trajectories

The phase trajectories for the nonlinear approximate system (25), (26) are determined by

$$\frac{dy}{dx} = \frac{y}{2x} - C_1 + \frac{x}{y}\ C_2\,, \tag{28}$$

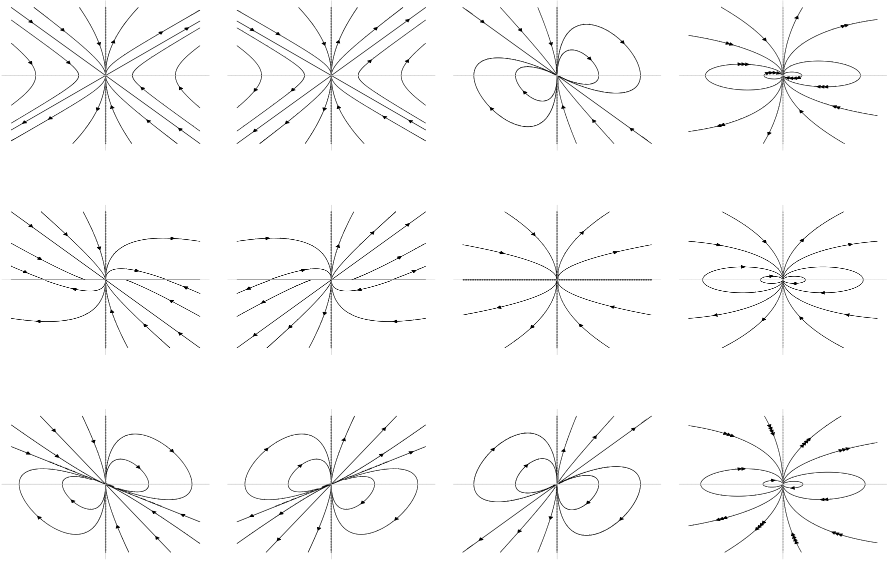
and its solutions depend on the sign of $C_1^2 + 2C_2 \equiv C$:

$$|x|K = \left|\frac{1}{2}y^2 + C_1 yx - C_2 x^2\right| \exp(-C_1 f(u))\,, \quad u \equiv \frac{y}{x}\,, \tag{29}$$

$$\begin{aligned} f(u) &= \frac{1}{\sqrt{C}} \ln\left|\frac{u + C_1 - \sqrt{C}}{u + C_1 + \sqrt{C}}\right| && \text{if} \quad C > 0\,, \\ &= -\frac{2}{u + C_1} && \text{if} \quad C = 0\,, \\ &= \frac{2}{\sqrt{|C|}}\left(\arctan\frac{u + C_1}{\sqrt{|C|}} + n\pi\right) && \text{if} \quad C < 0\,. \end{aligned} \tag{30}$$

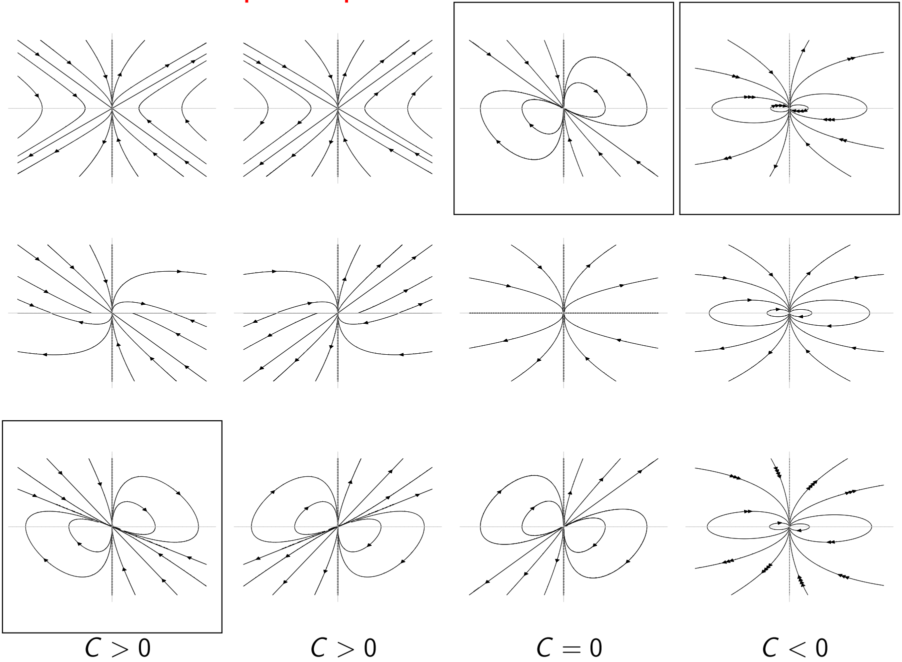
JKS 2010a

Classification of phase portraits



$C > 0$ | $C > 0$ | $C = 0$ | $C < 0$

Classification of phase portraits



Remarks

- We can argue that the topology of trajectories in the nonlinear approximation is representative of those of the full system, therefore should take the nonlinear approximation seriously.
- Typically there are many trajectories passing through the GR point either once or multiple times.
- In the end, only if

$$\frac{d}{d\Psi}\left(\frac{1}{2\omega(\Psi)+3}\right)\bigg|_{\Psi_\star}\left(2V(\Psi)-\frac{dV(\Psi)}{d\Psi}\Psi\right)\bigg|_{\Psi_\star}<0 \qquad (31)$$

  does the GR point function as an asymptotic attractor for the flow of all trajectories in the vicinity.

Time solutions

Can express these solutions also in terms of cosmological time:

$$\begin{aligned}
\pm x &= e^{-C_1 t}\left[M_1 e^{\frac{1}{2}t\sqrt{C}} - M_2 e^{-\frac{1}{2}t\sqrt{C}}\right]^2, \qquad \text{if} \quad C > 0\,, \quad (32)\\
&= e^{-C_1 t}\left[e^{\frac{1}{2}C_1 t_1} t - M_2\right]^2, \qquad \text{if} \quad C = 0\,, \quad (33)\\
&= e^{-C_1 t}\left[N_1 \sin(\frac{1}{2}t\sqrt{|C|}) - N_2 \cos(\frac{1}{2}t\sqrt{|C|})\right]^2, \text{if} \quad C < 0 \quad (34)
\end{aligned}$$

where M_1, M_2, t_1, N_1, N_2 are constants of integration (determined by initial conditions).

Expansion

Via Friedmann equation can express $H(x(t))$, $\dot{H}(x(t))$ and

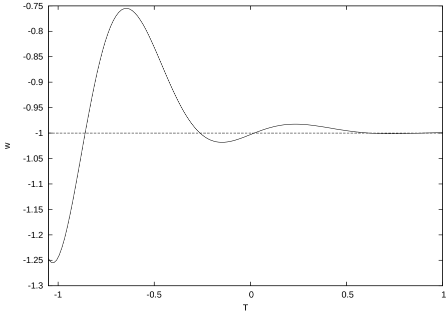
$$\mathrm{w} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \frac{1}{C_1^2\Psi_\star}\left[\frac{3}{2}\left(1 + \frac{1}{\Psi_\star A_\star}\right)\frac{\dot{x}^2}{x} - 4C_1\dot{x} + 3C_2 x\right] \tag{35}$$

The attractor solutions converge to de Sitter.

It is possible to have solutions which have oscillating or not-oscillating w, which are crossing the phantom divide ($w = -1$), and not crossing the phantom divide.

Classification JKS 2010b.

Example of oscillating dark energy



- Take $\omega(\Psi) = \frac{\Psi}{2(1-\Psi)}$, $\kappa^2 V(\Psi) = V_0 e^{3(1-\Psi)}$, the "GR point" is at $\Psi_\star = 1$.
- Initial conditions satisfy solar System bounds.
- Oscillations of w measured in the units of the analogue of Hubble time, $T = H_\star\, t = \frac{C_1}{3}\, t$.

Summary

- We have found and characterized the fixed points of STG cosmology in the case when potential dominates over cosmological matter density,
- in particular we have also found the general analytic form of solutions around the fixed points.
- This can be applied to cosmological expansion: can tell whether the solutions of any particular theory have oscillating, phantom crossing etc behavior.

- The analysis in the case of matter domination should be refined by carefully dealing with the indeterminacy in the equations.
- Next step, if possible: cross-over from matter domination to potential domination.

- Rely upon the attractor mechanism: instead of scanning the full phase space range of all theories, focus upon the vicinity of certain points which are favored by cosmological dynamics.
- Selection principle: only those theories and models are viable, which possess attractive fixed points, around where solutions satisfy observational constraints.