Application of Hidden Symmetries to Black Holes Physics

Valeri P. Frolov

University of Alberta

http://neondude.uw.hu/
esher_mosaic_ii.jpg

NEB 14 Recent Developments in Gravity
June 8-11, 2010 Ioannina, Greece

Main focus: Properties of four and higher dimensional rotating black holes

Spherical topology of the horizon

Either asymptotically flat (vacuum) or asympt. (A)deSitter (with cosmological constant)

Particle motion (mainly geodesics)

Field propagation

Key words: Hidden symmetries

Complete integrability

Separation of variables

Phase Space

Phase space: M^{2m}, Ω, H;

Ω is a closed non-degenerate 2-form (symplectic form):

dΩ=0 (Ω=dα)

Hamiltonian H is a scalar function on the symplectic manifold M^{2m}

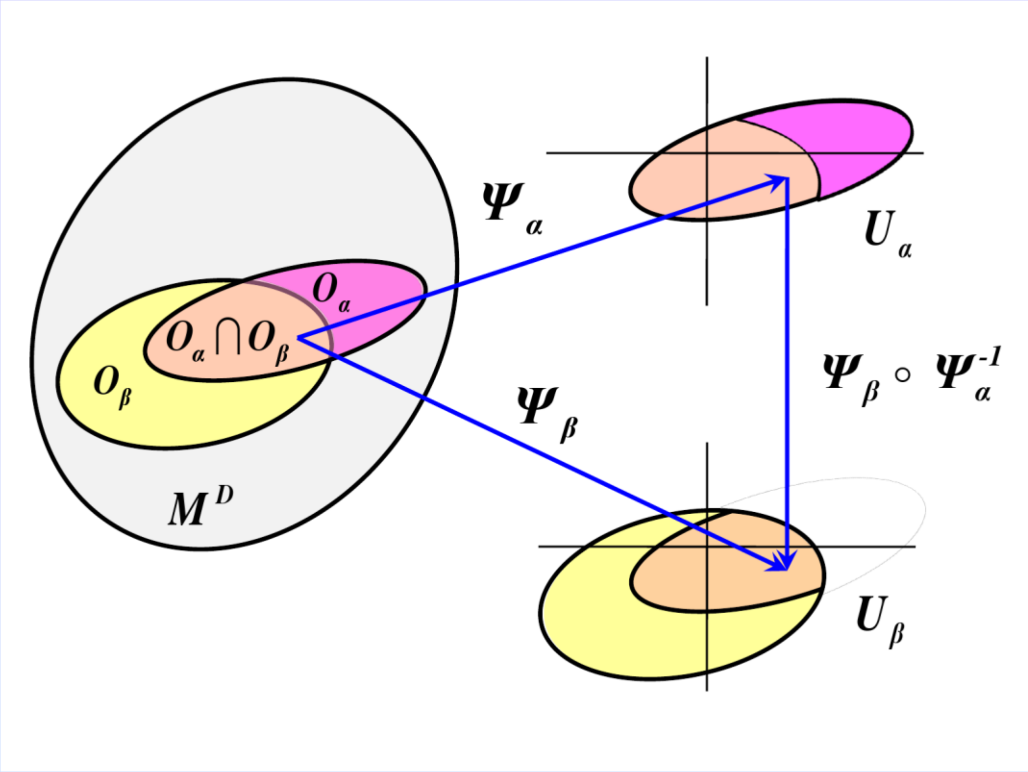
z^A are coordinates on M^{2m}

Poisson bracket $\{F,G\} = \Omega^{AB} F_{,A} G_{,B}$

$\eta^A = H_{,B}\Omega^{BA}$ is a generator of the Hamiltonian flow

Equation of motion is $\dot{z}^A = \eta^A$

One has $\dot{F} = \{H,F\}$



Darboux theorem: In the vicinity of any point it is always possible to choose canonical coordinates

$z^A = (p_1, \ldots, p_m, q_1, \ldots, q_m)$ in which $\Omega = \sum_{i=1}^{m} dp_i \wedge dq_i$

Integrability means: reducible to quadratures

Integrability is linked to `existence of constants of motion'

- How many constants of motion
- How precisely they are related
- How the phase space is foliated by their level sets

A system of differential equations is said to be integrable by quadratures if its solutions can be found after a finite number of steps involving algebraic operations and integration of given functions.

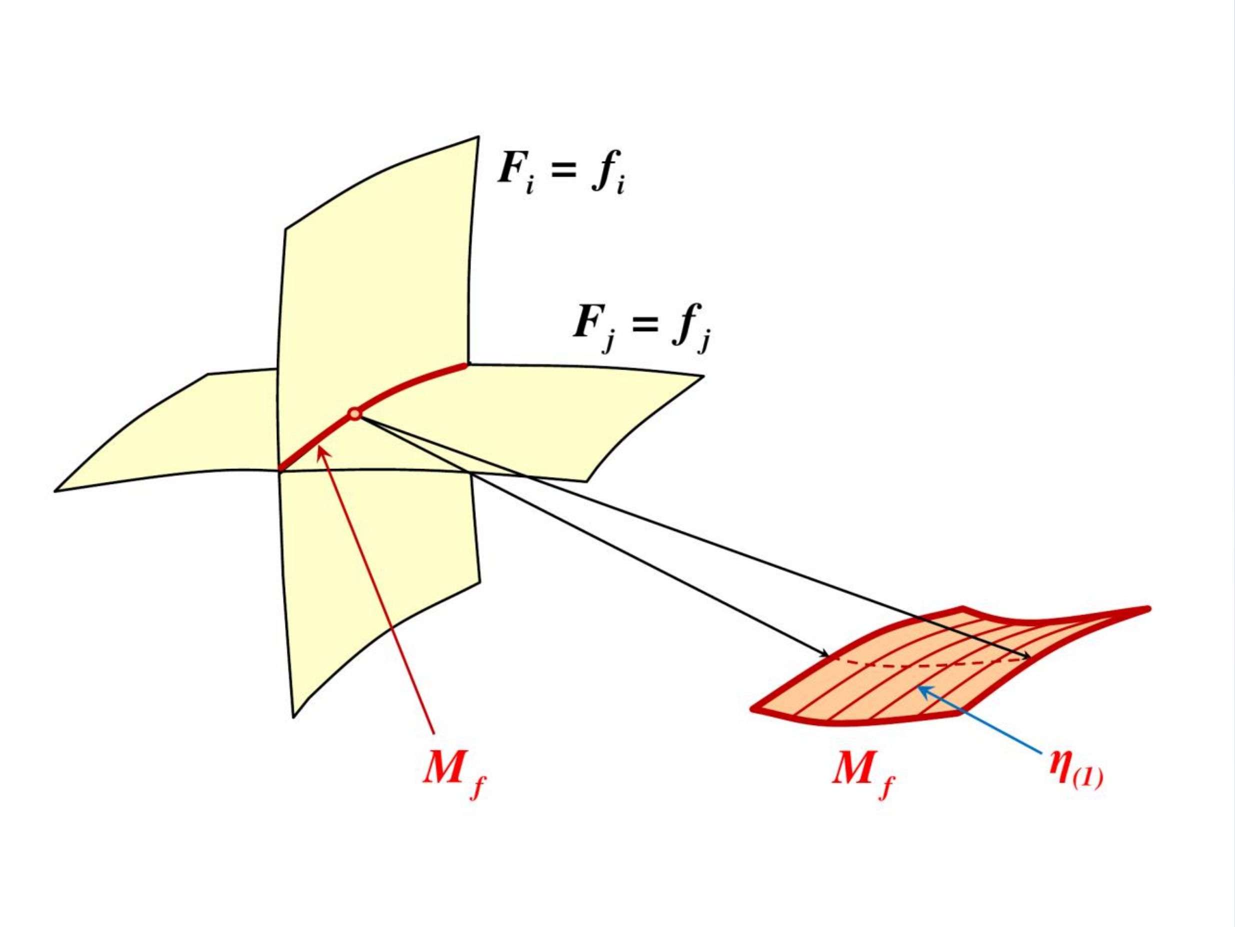
Liouville theorem (Bour, 1855;   Liouville, 1855):

If a Hamiltonian system with m degrees of freedom has m integrals of motion $F_1 = H, \ldots, F_m$ in involution, $\{F_i, F_j\} = 0,$ and functionally independent on the intersection of the levels sets of the m functions, $F_i = f_i$, the solutions of the corresponding Hamilton's equations can be found by quadratures.

"Miracle": Each of the integrals of motion "works twice"

Liouville Integrable System

The main idea behind Liouville's theorem is that the first integrals F_i can be used as local coordinates. The involution condition implies that the m vector fields, generated by F_i commute with each other and provide a choice of canonical coordinates. In these coordinates, the Hamiltonian is reduced to a sum of m decoupled Hamiltonians that can be integrated



$$X_i^{\ A} = \Omega^{AB} F_{i,B}$$

(1) X_i are tangent to M_f;

(2) $[X_i, X_j] = 0$

There exists a canonical transformation $(p_i, q_i) \to (F_{i,} \Psi_i)$

$$\Omega = \sum_{i=1}^{m} dp_i \wedge dq_i = \sum_{i=1}^{m} dF_i \wedge d\Psi_i$$

To obtain Ψ_i 2 operations are required:

(1) Find $p_i = p_i(f, q),$ and (2) calculate some integrals

The system in the new variables takes the form

$$\dot{F}_i = \{H, F_i\} = 0;$$

$$\dot{\Psi}_i = \{H, \Psi_i\} = \frac{\partial H}{\partial F_i} = \delta_i^1;$$

$$F_i = const, \quad \Psi_i = a + bt$$

Integrability and chaotic motion are at the two ends of `properties' of a dynamical system. But integrability is exceptional, chaoticity is generic.

In all cases, integrability seems to be deeply related with some symmetry, which might be partially hidden: the existence of constants of motion reflects the symmetry.

Important known examples of integrable mechanical systems include:

(1) Motion in Euclidean space under central potential
(2) Motion in the two Newtonian fixed centers
(3) Geodesics on an ellipsoid (Jacobi, 1838)
(4) Motion of a rigid body about a fixed point (several cases)
(Euler, Lagrange, Kowalevski)
(5) Neumann model

The Neumann model

$$L = \sum_{k=1}^{N} \tfrac{1}{2}(\dot{x}_k^2 - a_k x_k^2) + \tfrac{1}{2}\Lambda \sum_{k=1}^{N} (x_k^2 - 1)$$

Particle motion in GR

Phase space in GR: (p_μ, x^ν) are canonical coordinates:

$$\Omega = \sum_\mu dp_\mu \wedge dx^\mu. \quad \text{Hamiltonian } H = \tfrac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu$$

Equations of motion:

$$\dot{x}^\mu = \{H, x^\mu\} = g^{\mu\nu} p_\nu, \quad \dot{p}_\mu = \{H, p_\mu\} = -\tfrac{1}{2} g^{\nu\lambda}{}_{,\mu} p_\nu p_\lambda$$

are equivalent to the geodesic equation $p^\nu p_{\mu;\nu} = 0.$

Consider a special monom on the phase space

$\mathbf{K} = K_{\mu_1 \dots \mu_s} p^{\mu_1} \dots p^{\mu_s}$. A condition that it is a first integral of motion implies: $K_{(\mu_1 \dots \mu_s ; \nu)}=0$, i.e. $K_{\mu_1 \dots \mu_s}$ is a Killing tensor.

Remark: $g_{\mu\nu}$ is a trivial Killing tensor of rank 2.

The Poisson bracket $\{\mathbf{K}_1, \mathbf{K}_2\} \Rightarrow [K_1, K_2] = K_1 \overleftrightarrow{\partial} K_2$.

The first integrals of motion $\mathbf{K}_1$ and $\mathbf{K}_2$ are in involution when $[K_1, K_2] = 0$.

If there exist m non-degenerate functionally independent Killing tensors in involution then the geodesic equations in m dimensions are completely integrable.

New physically interesting wide class of completely integrable systems

Geodesic motion in the gravitational field of 4 and higher dimensional rotating black holes with spherical topology of the horizon (with `NUT' parameters) in the asymptotically flat or (A)dS

Separation of variables in HJ eqs

For the Hamiltonian $H(P,Q)$, $P = p_1,\dots,p_m$, $Q = q_1,\dots,q_m$,
the Hamilton-Jacobi equation is $H(\partial_P S, Q) = 0.$

Suppose q_1 and $\partial_{q_1} S$ enter this equation as $\Phi_1(\partial_{q_1} S, q_1)$ then the variable q_1 can be separated:

$$S = S_1(q_1, C_1) + S'(q_2,\dots,q_m), \quad \Phi_1(\partial_{q_1} S, q_1) = C_1,$$

$$H_1(\partial_{q_2} S',\dots,q_2,\dots;C_1) = 0$$

Complete separation of variables:

$$S = S_1(q_1,C_1) + S_2(q_2,C_1,C_2) + \dots S_m(q_m,C_1,\dots,C_m)$$

The constants C_i generate first integrals on the phase space. When these integrals are independent and in involution the system is integrable in the Liouville sence.

Forms (=AStensor)

(1) External product: $\alpha_q \wedge \beta_p = (\alpha \wedge \beta)_{q+p}$

(2) Hodge dual: $*(\alpha_q) = (*\alpha)_{D-q}$

(3) External derivative: $d(\alpha_q) = (d\alpha)_{q+1}$

(4) Closed form: $d(\alpha_q) = 0$ (locally $\alpha_q = d(\beta_{q-1})$)

CKY=Conformal Killing-Yano tensor

$$k_{\mu_1\mu_2\ldots\mu_n} = k_{[\mu_1\mu_2\ldots\mu_n]}, \quad \tilde{k}_{\mu_2\ldots\mu_n} \equiv \nabla^{\mu_1} k_{\mu_1\mu_2\ldots\mu_n}$$

$$\nabla_{(\mu_1} k_{\mu_2)\mu_3\ldots\mu_{n+1}} = g_{\mu_1\mu_2}\tilde{k}_{\mu_3\ldots\mu_{n+1}} - (n-1)g_{[\mu_3(\mu_1}\tilde{k}_{\mu_2)\ldots\mu_{n+1}]}$$

If $\tilde{k}$ vanishes f=k is a Killing-Yano tensor

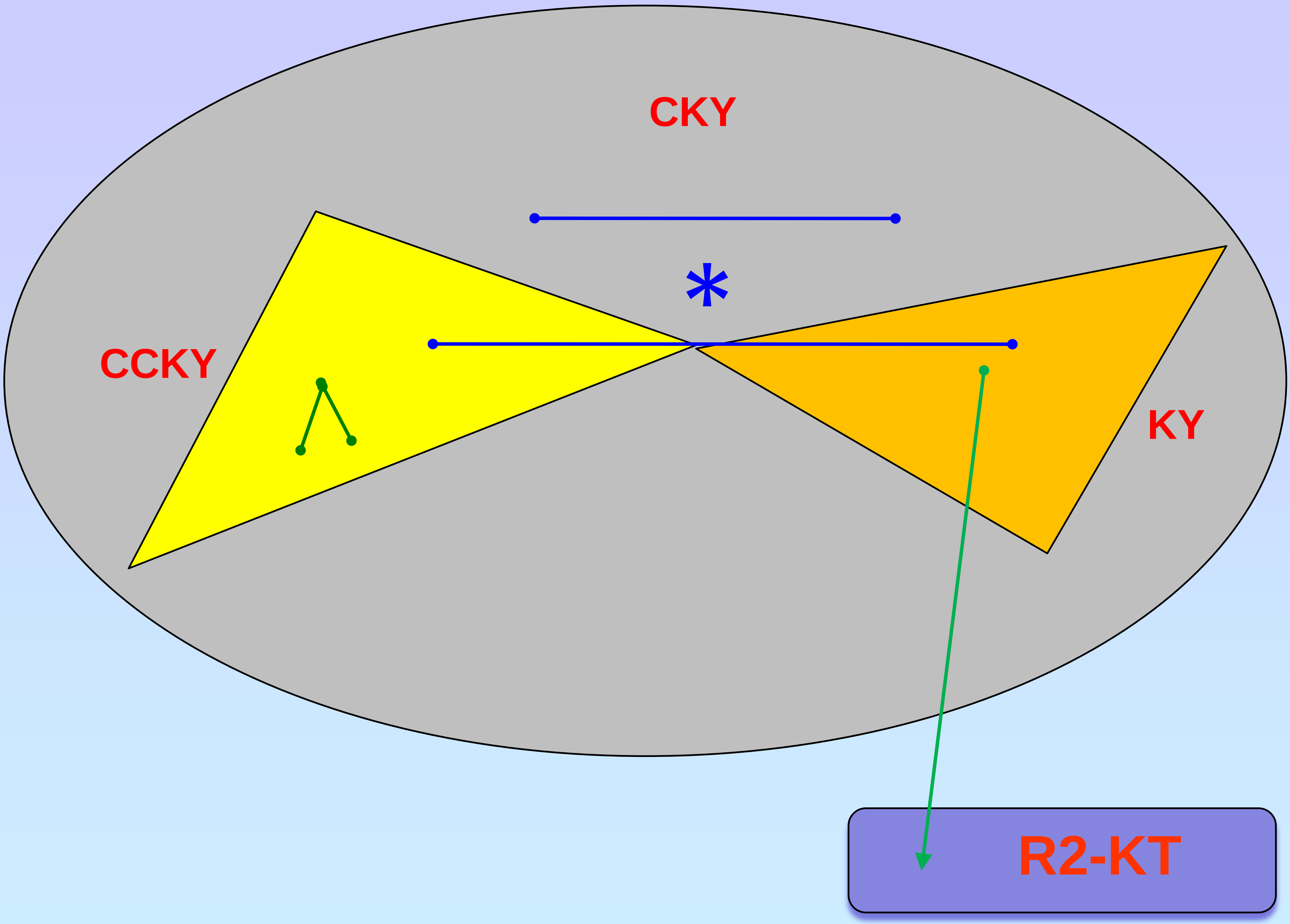
$K_{\mu\nu} = f_{\mu\mu_2\ldots\mu_n} f_{\nu}{}^{\mu_2\ldots\mu_n}$ is the Killing tensor

Properties of CKY tensor

Hodge dual of CKY tensor is CKY tensor

Hodge dual of closed CKY tensor is KY tensor

External product of two closed CKY tensors is a closed CKY tensor


CKY
*
CCKY
KY
R2-KT

Principal Killing-Yano tensor

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \quad (*)$$

$$\nabla_{[a} h_{bc]} = 0, \quad \xi_a = \frac{1}{D-1}\nabla^b h_{ba}$$

PKY tensor is a closed non-degenerate (matrix rank 2n) 2-form obeying (*)

ξ^a is a primary Killing vector (off-shell!!)

Killing-Yano Tower

Killing-Yano Tower

$$h \;\Rightarrow\; h^{\wedge j} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{j\ times}$$

$$k_j = {*h^{\wedge j}} \qquad\qquad K^j = k_j \cdot k_j$$

$$\xi_j = K_j \cdot \xi$$

Total number of conserved quantities

$$\underset{KV}{(n+\varepsilon)} + \underset{KT}{(n-1)} + \underset{g}{1} = 2n + \varepsilon = D$$

Existence of the Principal CCKY tensor in the most general known solution for higher dimensional rotational Kerr-NUT-(A)dS black hole metric was discovered in:

V. F., D.Kubiznak, Phys.Rev.Lett. 98, 011101 (2007); gr-qc/0605058

D. Kubiznak, V. F., Class.Quant.Grav.24:F1-F6 (2007); gr-qc/0610144

Constructed KY tower produces a set of D non-degenerate, functionally independent Killing integrals of motion in the involution

P. Krtous, D. Kubiznak, D. N. Page, V. F., JHEP 0702:004 (2007)

D. N. Page, D. Kubiznak, M. Vasudevan, P. Krtous, Phys.Rev.Lett. (2007)

P. Krtous, D. Kubiznak, D. N. Page, M. Vasudevan, PRD76:084034 (2007);

Metrics which admit Principal CCKY tensor allow complete description

$$g_{ab} = \sum_{\mu} (e_a^{\mu} e_b^{\mu} + e_a^{\hat{\mu}} e_b^{\hat{\mu}}) + \varepsilon e_a^{n+1} e_b^{n+1},$$

$$e^{\mu} = \frac{1}{\sqrt{Q_{\mu}}} dx_{\mu}, \quad e^{\hat{\mu}} = \sqrt{Q_{\mu}} \sum_{i=0}^{n-1} A_{\mu}^{(i)} d\psi_i$$

$$Q_{\mu} = \frac{X_{\mu}}{U_{\mu}}, \quad , \quad U_{\mu} = \prod_{\nu \neq \mu} (x_{\nu}^2 - x_{\mu}^2), \quad X_{\mu} = X_{\mu}(x_{\mu})$$

$$\prod_{\nu=1}^{n} (1 + t x_{\nu}^2) = \sum_{j=0}^{n} t^j A^{(j)}, \qquad (1 + t x_{\mu}^2)^{-1} \prod_{\nu=1}^{n} (1 + t x_{\nu}^2) = \sum_{k=0}^{n-1} t^k A_{\mu}^{(k)}.$$

Houri, Oota, and Yasui [PLB (2007); JP A41 (2008)] proved this result under additional assumptions: $L_{\xi} g = 0$ and $L_{\xi} h = 0$. More recently Krtous, V.F., Kubiznak [arXiv:0804.4705 (2008)] and Houri, Oota, and Yasui [arXiv:0805.3877 (2008)] proved this without additional assumptions.

Canonical Coordinates

$$h \cdot m_{\pm}^{\mu} = \mp i x^{\mu} m_{\pm}^{\mu}$$

A non-degenerate 2-form h has n independent eigenvalues

n essential coordinates x^{μ} and $n+\varepsilon$ Killing coordinates ψ_j are used as canonical coordinates

D=2n+ε

Principal CCKY tensor

Non-degeneracy:

(1) Eigen-spaces of h are 2-dimensional

(2) x_μ are functionally independent in some domain

(they can be used as essential coordinates)

(1) is proved by Houri, Oota and Yasui e-print **arXiv:0805.3877**

(2) Case when some of eigenvalues are constant studied in
Houri, Oota and Yasui **Phys.Lett.B666:391-394,2008**.
e-Print: **arXiv:0805.0838**

On-Shell Result

A solution of the vacuum Einstein equations with the cosmological constant which admits a (non-degenerate) principal CKY tensor coincides with the Kerr-NUT-(A)dS spacetime.

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k}$$

Kerr-NUT-(A)dS spacetime is the most general BH solution obtained by Chen, Lu, and Pope [CQG (2006)]; See also Oota and Yasui [PL B659 (2008)]

"General Kerr-NUT-AdS metrics in all dimensions“, Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$n=\left[D/2\right],\;\; D=2n+\varepsilon$$

$$R_{\mu\nu}=(D-1)\lambda g_{\mu\nu}$$

$\lambda,\, M - mass,\, a_k - (n-1+\varepsilon)\; rotation\;\; parameters,$

$M_\alpha - (n-1-\varepsilon)\; `NUT'\;\; parameters$

$Total\;\#\,of\;\; parameters\; is\; D-\varepsilon$

Separability of the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial \lambda} + g^{ab}\, \partial_a S\, \partial_b S = 0$$

$$S = -w\lambda + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{m} \Psi_k \psi_k$$

$$(S_\mu{}')^2 = -\frac{1}{X_\mu^2}\left(\sum_{k=0}^{m} (-x_\mu^2)^{n-1-k}\, \Psi_k\right)^2 + \frac{1}{X_\mu}\sum_{k=0}^{m} \tilde{c}_k (-x_\mu^2)^{n-1-k}$$

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007); hep-th/0611245

Separability of the Klein–Gordon equation

$$(\Box - \mu^2)\Phi = 0$$

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{k=0}^{m} e^{i\Psi_k \psi_k}.$$

,

$$(X_\mu R_\mu)' + \varepsilon \frac{X_\mu}{x_\mu} R_\mu - \frac{R_\mu}{X_\mu}\Big(\sum_{k=0}^{m} (-x_\mu^2)^{n-1-k}\Psi_k\Big)^2 - \sum_{k=0}^{m} b_k (-x_\mu^2)^{n-1-k} R_\mu = 0$$

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007); hep-th/0611245

Weakly charged higher dimensional rotating black holes

$$H=\frac{1}{2}g^{ab}(p_a-qA_a)(p_b-qA_b) \qquad -\mu^2=g^{ab}\left[(\frac{\partial S}{\partial x^a}-qA_a)(\frac{\partial S}{\partial x^b}-qA_b)\right]$$

$$\left[g^{ab}(\nabla_a-iqA_a)(\nabla_b-iqA_b)-\mu^2\right]\Phi=0$$

$$F_{ab}^{\ ;b}=0 \quad \leftrightarrow \quad A^a_{\ ,a}=0 \quad \leftrightarrow \quad A^{a;b}_{\ ,b}=0$$

$$\xi^{a;b}_{\ ;b}=0 \quad \leftrightarrow \quad A_a=Q\xi_a \ \text{(in Ricci flat)}$$

$$g^{ab}\frac{\partial S}{\partial x^a}\frac{\partial S}{\partial x^b}+M^2=0, \quad [\Box-M^2]\Phi=0,$$

$$M^2=\mu^2-2e\Psi_0+e^2\xi^2_{(0)}$$

For a primary Killing vector field one again has a complete separation of variables

[V.F. 2010 (under preparation)]

Notes on Parallel Transport

Case 1: Parallel transport along timelike geodesics

Let u^a be a vector of velocity and h_{ab} be a PCKYT.

$P_a^b = \delta_a^b + u_a u^b$ is a projector to the plane orthogonal to u^a.

Denote $F_{ab} = P_a^c P_b^d h_{cd} = h_{ab} + u_a u^c h_{cb} + h_{ac} u^c u_b$

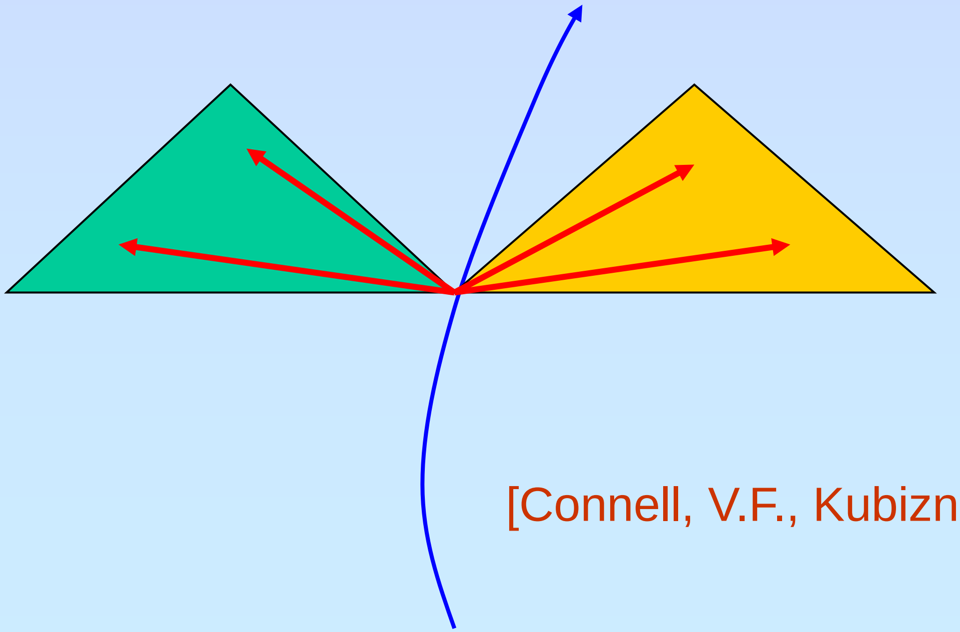
Lemma (Page): F_{ab} is parallel propagated along a geodesic:

$$\nabla_u F_{ab} = 0$$

Proof: We use the definition of the PCKYT

$$\nabla_u h_{ab} = u_a \xi_b - \xi_a u_b$$

Suppose h_{ab} is a non-degenerate, then for a generic geodesic eigen spaces of F_{ab} with non-vanishing eigen values are two dimensional. These 2D eigen spaces are parallel propagated. Thus a problem reduces to finding a parallel propagated basis in 2D spaces. They can be obtained from initially chosen basis by 2D rotations. The ODE for the angle of rotation can be solved by a separation of variables.



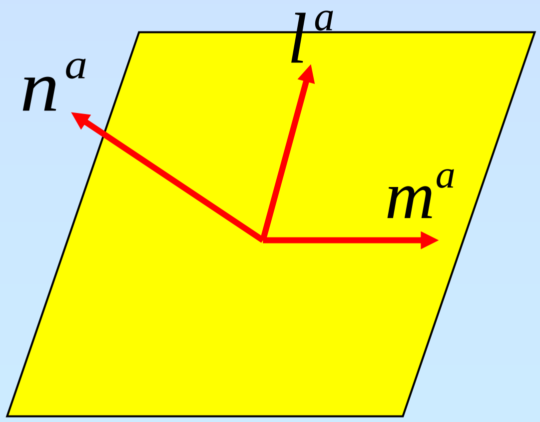
[Connell, V.F., Kubiznak, PRD 78, 024042 (2008)]

Case 2: Parallel transport along null geodesics

Let l^a be a tangent vector to a null geodesic and k^a be a parallel propagated vector obeying the condition $l^a k_a = 0$.

Then the vector $w^a = k_b h^{ba} + \beta l^a$ is parallel propagated, provided $\dot{\beta} = k_a \xi^a$.

This procedure allows one to construct 2 more parallel propagated vectors m^a and n^a, starting with l^a.



We introduce a projector $P_{ab} = g_{ab} + 2l_{(a}n_{b)}$, and $F_{ab} = P_a^c P_b^d h_{cd}$.

One has: $\nabla_l F_{ab} = P_a^c P_b^d \nabla_l h_{cd} = 2 P_a^c P_b^d l_{[c}\xi_{d]} = 0.$

Thus F_{ab} is parallel propaged along a null geodesic. We use rotations in its 2D eigen spaces to construct a parallel propagated basis.

[Kubiznak,V.F.,Krtous, Connell, PRD 79, 024018 (2009)]

Further Developments

Separability of the massive Dirac equation in the Kerr-NUT-(A)dS spacetime [Oota and Yasui, Phys. Lett. B 659, 688 (2008)]

Stationary string equations in the Kerr-NUT-(A)dS spacetime are completely integrable.
[D. Kubiznak, V. F., JHEP 0802:007,2008; arXiv:0711.2300]

Solving equations of the parallel transport along geodesics [P. Connell, V. F., D. Kubiznak, PRD,78, 024042 (2008); arXiv:0803.3259; D. Kubiznak, V. F., P. Connell, arXiv:0811.0012 (2008)]

Einstein spaces with degenerate closed conformal KY tensor [Houri, Oota and Yasui **Phys.Lett.B666:391-394,2008**. e-Print: **arXiv:0805.0838]**

Separability of Gravitational Perturbation in Generalized Kerr-NUT-de Sitter Spacetime
[Oota, Yasui, arXiv:0812.1623]

On the supersymmetric limit of Kerr-NUT-AdS metrics [Kubiznak, arXiv:0902.1999]

GENERALIZED KILLING-YANO TENSORS

[Kubiznak, Kunduri, and Yasui, 0905.0722 (2009)]

Minimally gauged supergravity (5D EM with Chern-Simons term):

$$L = *(R+\Lambda) - \tfrac{1}{2} F \wedge *F + \tfrac{1}{3\sqrt{3}} F \wedge F \wedge A,$$

$$dF = 0, \quad d*F - \tfrac{1}{\sqrt{3}} F \wedge F = 0,$$

$$R_{ab} + \tfrac{1}{3}\Lambda g_{ab} = \tfrac{1}{2}\left(F_{ac}F_b{}^c - \tfrac{1}{6} g_{ab} F^2\right)$$

Torsion: $T = \frac{1}{\sqrt{3}} * F$

$$\nabla^T_c h_{ab} = \nabla_c h_{ab} - \tfrac{1}{\sqrt{3}} (*F)_{cd[a} h^d{}_{b]} = 2 g_{c[a} \xi_{b]},$$

$$K_{ab} = (*h)_{acd} (*h)_b{}^{cd} = h_{ac} h_b{}^c - \tfrac{1}{2} g_{ab} h^2$$

Application: Chong, Cvetic, Lu, Pope [PRL, 95,161301,2005]

Note: This is type I metric.

Summary

The most general spacetime admitting the PCKY tensor is described by the canonical metric. It has the following properties:

- It is of the algebraic type D
- It allows a separation of variables for the Hamilton-Jacoby, Klein-Gordon, Dirac, tensorial gravitational perturbations, and stationary string equations
- The geodesic motion in such a spacetime is completely integrable. The problem of finding parallel-propagated frames reduces to a set of the first order ODE. This is a new interesting example of completely integrable system.
- When the Einstein equations with the cosmological constant are imposed the canonical metric becomes the Kerr-NUT-(A)dS spacetime
- Possible generalizations to degenerate PCKY tensor and non-vacuum STs