# Towards Canonical Quantum Gravity for Geometries Admitting Maximally Symmetric Two-dimensional Surfaces

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# Contents

# Preliminaries: Quantization of Finite Degrees of Freedom (1st class) Constraint Systems

- We assume a system described by a Hamiltonian of the form:

$$
\begin{aligned}
H &\equiv \mu X + \mu^i \chi_i \\
&= \mu \left( \frac{1}{2} G^{AB}(Q^\Gamma) P_A P_B + U^A(Q^\Gamma) P_A + V(Q^\Gamma) \right) \\
&\quad + \mu^i \, \phi_i^A(Q^\Gamma) P_A
\end{aligned}
$$

- Primary Constraints: $P_\mu \approx 0$, $P_{\mu^i} \approx 0$
- Secondary Constraints: $X \approx 0$, $\chi_i \approx 0$
- The constraints are assumed to be first class:

$$
\{X, X\} = 0, \qquad \{X, \chi_i\} = X C_i + C_i^j \chi_j, \qquad \{\chi_i, \chi_j\} = C_{ij}^k \chi_k
$$

- The physical state of the system is unaffected by the "gauge" transformations generated by $(X, \chi_i)$, i.e. $\delta_i A = \{A, \chi_i\}$, $\delta_0 A = \{A, X\}$
  **but also** under the following three changes:

  (I) Mixing of the super-momenta with a non-singular matrix

  $$\bar{\chi}_i = \lambda_i^j(Q^\Gamma)\chi_j$$

  (II) Gauging of the super-Hamiltonian with the super-momenta

  $$\bar{X} = X + \kappa^{(Ai}(Q^\Gamma)\phi_i^{B)}(Q^\Gamma)P_A P_B + \sigma^i(Q^\Gamma)\phi_i^A(Q^\Gamma)P_A$$

  (III) Scaling of the super-Hamiltonian

  $$\bar{X} = \tau^2(Q^\Gamma)X$$

- The whole quantization scheme must be independent of actions (I), (II), (III). This is achieved by:
  - (1) Realize the linear operator constraint conditions with the momentum operators to the right

    $$\hat{\chi}_i \Psi = 0 \leftrightarrow \phi_i^A(Q^\Gamma)\frac{\partial \Psi(Q^\Gamma)}{\partial Q^A} = 0,$$

    maintaining the geometrical meaning of the linear constraints and producing the $M - N$ independent solutions $q^\alpha(Q^\Gamma)$, $\alpha = 1, 2, \ldots, M - N$ called physical variables.

  - (2) Define the induced structure $g^{\alpha\beta} \equiv G^{AB}\dfrac{\partial q^\alpha}{\partial Q^A}\dfrac{\partial q^\beta}{\partial Q^B}$ and realize the quadratic in momenta part of $X$ as the conformal Laplace-Beltrami operator based on $g_{\alpha\beta}$.

- $g^{\alpha\beta}(q^\gamma)$ by virtue of the 1st class algebra.

# Possible metrics and Hamiltonian Formulation

Two-dimensional spaces of positive, zero and negative constant curvature:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \quad ds^2 = d\theta^2 + \theta^2\, d\phi^2, \quad ds^2 = d\theta^2 + \sinh^2\theta\, d\phi^2$$

The corresponding (maximal) symmetry groups are generated by the following KVF's:

$$\xi_1 = \frac{\partial}{\partial\phi}, \qquad \xi_2 = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}$$

$$\xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

$$\xi_1 = \frac{\partial}{\partial \phi}, \qquad \xi_2 = -\cos \phi \frac{\partial}{\partial \theta} + \frac{\sin \phi}{\theta} \frac{\partial}{\partial \phi}$$

$$\xi_3 = \sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \phi}{\theta} \frac{\partial}{\partial \phi}$$

$$\xi_1 = \frac{\partial}{\partial \phi}, \qquad \xi_2 = -\cos \phi \frac{\partial}{\partial \theta} + \coth \theta \sin \phi \frac{\partial}{\partial \phi}$$

$$\xi_3 = \sin \phi \frac{\partial}{\partial \theta} + \coth \theta \cos \phi \frac{\partial}{\partial \phi}$$

Reduced line element:

$$ds^2 = \left( -\alpha(t,r)^2 + \frac{\beta(t,r)^2}{\gamma(t,r)^2} \right) dt^2 + 2\,\beta(t,r)\,dt\,dr + \gamma(t,r)^2\,dr^2$$
$$+ \psi(t,r)^2\,d\theta^2 + \psi(t,r)^2\,f(\theta)^2\,d\phi^2$$

where:

- $f(\theta) = \sin\theta$ : spherical symmetry,
- $f(\theta) = \theta$ : plane symmetry,
- $f(\theta) = \sinh\theta$ : GBL symmetry.

- We define the vectors

$$\eta^I = \frac{1}{\alpha(t,r)}(1, -\frac{\beta(t,r)}{\gamma(t,r)^2}, 0, 0), \qquad F^I = \eta^J_{;J}\, \eta^I - \eta^I_{;J}\, \eta^J,$$

- Utilizing the Gauss-Codazzi equation, yields an action quadratic in the velocities,

$$I = \int d^4x \sqrt{-g}(R - 2\Lambda - 2\, F^I_{;I})$$

- Dirac algorithm results in the primary constraints

$$P_\alpha \equiv \frac{\delta L}{\delta \dot{\alpha}} \approx 0, \qquad P^\beta \equiv \frac{\delta L}{\delta \dot{\beta}} \approx 0$$

- and the Hamiltonian

$$H = \int \left( N^o\, \mathcal{H}_o + N^i\, \mathcal{H}_i \right) dr,$$

where

$$N^o = \alpha(t,r), \quad N^1 = \frac{\beta(t,r)}{\gamma(t,r)^2}, \quad N^2 = 0, \quad N^3 = 0$$

$$
\begin{aligned}
\mathcal{H}_o &= \frac{1}{2} G^{\alpha\beta} \pi_\alpha \pi_\beta + V, \\
\mathcal{H}_1 &= -\gamma \pi'_\gamma + \psi' \pi_\psi, \qquad \mathcal{H}_2 = 0, \qquad \mathcal{H}_3 = 0,
\end{aligned}
$$

and $\{\alpha, \beta\}$ take the values $\{\gamma, \psi\}$ while $' = \dfrac{d}{d\,r}$.

- Reduced Wheeler-deWitt super-metric $G^{\alpha\beta}$:

$$G^{\alpha\beta} = \begin{pmatrix} \dfrac{\gamma}{4\,\psi^2} & -\dfrac{1}{4\,\psi} \\[3mm] -\dfrac{1}{4\,\psi} & 0 \end{pmatrix},$$

while the potential $V$ is

$$V = -2\,\epsilon\,\gamma + 2\Lambda\gamma\,\psi^2 - 2\frac{\psi'^{\,2}}{\gamma} + 4\left(\frac{\psi\,\psi'}{\gamma}\right)'$$

with $\epsilon = \{1, 0, -1\}$ for positive, zero or negative constant curvature, respectively.

- The requirement for preservation, in time, of the primary constraints leads to the secondary constraints:

$$\mathcal{H}_o \approx 0, \qquad\qquad \mathcal{H}_1 \approx 0$$

- Open Poisson bracket algebra of these constraints:

$$\{\mathcal{H}_o(r), \mathcal{H}_o(\tilde{r})\} = \left[\frac{1}{\gamma^2(r)}\mathcal{H}_1(r) + \frac{1}{\gamma^2(\tilde{r})}\mathcal{H}_1(\tilde{r})\right]\delta'(r,\tilde{r}),$$

$$\{\mathcal{H}_1(r), \mathcal{H}_o(\tilde{r})\} = \mathcal{H}_o(r)\delta'(r,\tilde{r}),$$

$$\{\mathcal{H}_1(r), \mathcal{H}_1(\tilde{r})\} = \mathcal{H}_1(r)\delta'(r,\tilde{r}) - \mathcal{H}_1(\tilde{r})\delta(r,\tilde{r})'.$$

- Under general changes $r \to \tilde{r} = h(r)$, it follows:

$$\tilde{\gamma}(\tilde{r}) = \gamma(r) \frac{d\,r}{d\,\tilde{r}}, \qquad \tilde{\psi}(\tilde{r}) = \psi(r), \qquad \frac{d\,\tilde{\psi}(\tilde{r})}{d\,\tilde{r}} = \frac{d\,\psi(r)}{d\,r} \frac{d\,r}{d\,\tilde{r}},$$

- With the infinitesimal transformation $r \to \tilde{r} = r - \eta(r)$, the corresponding changes induced on the basic fields are:

$$\delta\,\gamma(r) = (\gamma(r)\,\eta(r))', \qquad\qquad \delta\,\psi(r) = \psi'(r)\,\eta(r)$$

(one-dimensional analogue of the appropriate Lie derivatives).

- Action of $\mathcal{H}_1$ on the basic configuration space variables: Generator of spatial diffeomorphisms, i.e.

$$\left\{ \gamma(r), \int d\tilde{r}\,\eta(\tilde{r})\,\mathcal{H}_1(\tilde{r}) \right\} = (\gamma(r)\,\eta(r))',$$

$$\left\{ \psi(r), \int d\tilde{r}\,\eta(\tilde{r})\,\mathcal{H}_1(\tilde{r}) \right\} = \psi'(r)\,\eta(r).$$

# Canonical Quantization

- Realize classical momenta as functional derivatives with respect to their corresponding conjugate fields:

$$\hat{\pi}_\gamma(r) = -i\,\frac{\delta}{\delta\,\gamma(r)}, \qquad \hat{\pi}_\psi(r) = -i\,\frac{\delta}{\delta\,\psi(r)}.$$

- Decide on initial state of space vectors:

$$\hat{\pi}_\gamma(r)\gamma(\tilde{r})^2 = -2i\gamma(\tilde{r})\delta(\tilde{r},r).$$

- Choose as initial collection of states all smooth functionals of the configuration variables $\gamma(r), \psi(r)$ and their derivatives of any order:

$$\hat{\pi}_\gamma(r)\int d\tilde{r}\gamma(\tilde{r})^2 = -2i\int d\tilde{r}\gamma(\tilde{r})\delta(\tilde{r},r) = -2i\gamma(r)$$

But

$$\hat{\pi}_\gamma(r)\,\hat{\pi}_\gamma(r)\int d\tilde{r}\gamma(\tilde{r})^2 = \hat{\pi}_\gamma(r)(-2i\int d\tilde{r}\gamma(\tilde{r})\delta(\tilde{r},r)) = \hat{\pi}_\gamma(r)(-2i\gamma(r))$$

$$= -2\delta(0)$$

$$\hat{\pi}_\gamma(r)\int d\tilde{r}\gamma''(\tilde{r})^2 = -2i\int d\tilde{r}\gamma''(\tilde{r})\delta''(\tilde{r},r) = -2i\gamma^{(4)}(r)$$

Generalizing the H-Kuchař quantization yields a quantization scheme of our system which:

(a) avoids the occurrence of $\delta(0)$'s,

(b) reveals the value $n = 1$ as the only *natural* (i.e. without *ad-hoc* cut-offs) possibility to obtain a closed space of state vectors,

(c) extracts a finite-dimensional Wheeler-deWitt equation governing the quantum dynamics.

- **Step 1**: Define the quantum analogue of $\mathcal{H}_1(r) \approx 0$ as:

$$\hat{\mathcal{H}}_1(r)\Phi = 0 \leftrightarrow -\gamma(r)\,\big(\frac{\delta\,\Phi}{\delta\,\gamma(r)}\big)' + \psi'(r)\,\frac{\delta\,\Phi}{\delta\,\psi(r)} = 0.$$

- General Solutions:

$$\Phi \;=\; \int \gamma(\tilde{r})\,L\left(\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(n)}\right) d\tilde{r}$$

$$\Psi^{(0)} \;\equiv\; \psi(\tilde{r}), \quad \Psi^{(1)} \equiv \frac{\psi'(\tilde{r})}{\gamma(\tilde{r})}, \ldots, \Psi^{(n)} \equiv \frac{1}{\gamma(\tilde{r})}\frac{d}{d\tilde{r}}\left(\underbrace{\ldots}_{n-1}\psi(\tilde{r})\right)$$

where $L$ is any function of its arguments.

- **Step 2**: Define the equivalent of Kuchař's induced metric on the so far space of "physical" states Φ: analogues of Kuchař's physical variables $q^\alpha$.

- Consider **one** initial candidate of the above form.

- Generalize the partial to functional derivatives.

- ⇒ Induced metric:

$$g^{\Phi\Phi} = G^{\alpha\beta} \frac{\delta \Phi}{\delta x^\alpha} \frac{\delta \Phi}{\delta x^\beta}, \quad \text{where} \quad x^\alpha = \{\gamma, \psi\}$$

- Well-defined metric (it contains only first functional derivatives of the state vectors).

- Nevertheless, $g^{\Phi\Phi}$ *is* a local function and not a smooth functional. For the induced metric $g^{\Phi\Phi}$ to be composed out of the "physical" states annihilated by $\hat{\mathcal{H}}_1$, a correspondence between local functions and smooth functionals is necessary:

  **Assumption:** *We assume that, as part of the renormalization procedure, we are permitted to map local functions to their corresponding smeared expressions e.g.,* $\psi(r) \leftrightarrow \int d\tilde{r}\psi(\tilde{r})$.

- Let $\mathcal{F}$ be the space which contains all local functions, and define the equivalence relations

$$\sim: \{f_1(r) \sim f_2(\tilde{r}), \tilde{r} = g(r)\}, \quad \approx: \{h_1(r) \approx h_2(\tilde{r})\frac{d\,\tilde{r}}{d\,r}, \tilde{r} = g(r)\}$$

for scalars and densities respectively.

- In order to proceed with the generalization of Kuchař's method, we have to demand that:

  **Requirement:** $L\left(\Psi^{(0)}, \ldots, \Psi^{(n)}\right)$ *must be such that* $g^{\Phi\Phi}$ *becomes a general function, say* $F\left(\gamma(r)\, L(\Psi^{(0)}, \ldots, \Psi^{(n)})\right)$ *of the integrand of* $\Phi$, *so that it can be considered as a function of this state:*

  $g^{\Phi\Phi} \overset{\underset{Assumption}{}}{\equiv} F\left(\int \gamma(\tilde{r})\, L(\Psi^{(0)}, \ldots, \Psi^{(n)})d\tilde{r}\right) = F(\Phi).$

More precisely

$$g^{\Phi\Phi} = \ldots + 2\,G^{12}\,\frac{\delta\Phi}{\delta\gamma(r)}\,\frac{\delta\Phi}{\delta\psi(r)}\,,$$

where the functional derivatives are:

$$\frac{\delta\Phi}{\delta\psi} = \ldots + \int \gamma\,\frac{\partial L}{\partial\Psi^{(n)}}\,\frac{\delta\Psi^{(n)}}{\delta\psi}\,d\tilde{r}$$

$$= \ldots + \int \gamma\,\frac{\partial L}{\partial\Psi^{(n)}}\,\frac{1}{\gamma}\,\frac{d}{d\tilde{r}}\left(\underbrace{\ldots}_{n-1}\,\delta(r,\tilde{r})\right)d\tilde{r} =$$

$$= \ldots - \int \gamma\,\frac{\partial^2 L}{\partial\left(\Psi^{(n)}\right)^2}\,\Psi^{(n+1)}\,\frac{1}{\gamma}\,\frac{d}{d\tilde{r}}\left(\underbrace{\ldots}_{n-2}\,\delta(r,\tilde{r})\right)d\tilde{r} =$$

$$\vdots$$

$$= \ldots + (-1)^n\gamma\,\frac{\partial^2 L}{\partial\left(\Psi^{(n)}\right)^2}\,\Psi^{(2n)}$$

Similarly,

$$\frac{\delta \Phi}{\delta \gamma} = \ldots + (-1)^{n-1} \frac{\partial^2 L}{\partial \left( \Psi^{(n)} \right)^2} \; \Psi^{(2n-1)} \, \Psi^{(1)}.$$

Therefore

$$g^{\Phi\Phi} = \ldots - \frac{\gamma}{2\psi} (-1)^{2n-1} \left( \frac{\partial^2 L}{\partial \left( \Psi^{(n)} \right)^2} \right)^2 \Psi^{(1)} \, \Psi^{(2n-1)} \, \Psi^{(2n)},$$

where the ... stand for all other terms, not involving $\Psi^{(2n)}$.
According to the aforementioned **Requirement** we need this to
be a general function, say $F(\gamma L)$, and for this to happen the
coefficient of $\Psi^{(2n)}$ must vanish, i.e.

$$\frac{\partial^2 L}{\partial \left( \Psi^{(n)} \right)^2} = 0$$

$$\Leftrightarrow L = L_1 \left( \Psi^{(0)}, \ldots, \Psi^{(n-1)} \right) \Psi^{(n)} + L_2 \left( \Psi^{(0)}, \ldots, \Psi^{(n-1)} \right).$$

- For the local part of $\Phi$ only $L_2$ is important.
- All $\Psi^{(n)}$'s are suppressed from $L$ except when $n = 1$ since, upon elimination of the linear in $\Psi^{(2)}$ term we are left with a local function of $\Psi^{(1)}$, and thus the possibility arises to meet the **Requirement** by solving a differential equation for $L$. Hence, if

$$\Phi \equiv \int \gamma(\tilde{r}) L\left(\psi, \Psi^{(1)}\right) d\tilde{r},$$

through the definition

$$H \equiv L - \Psi^{(1)} \frac{\partial L}{\partial \Psi^{(1)}}$$

and by virtue of the **Assumption**, we obtain

$$g^{\Phi\Phi} = \frac{\gamma}{4\psi^2}\left(H^2 - 2\,\psi\,H\,\frac{\partial H}{\partial \psi} + 4\psi^2\Psi^{(1)}\frac{\partial}{\partial \psi}\int \frac{1}{2\psi\Psi^{(1)}}\,H\,\frac{\partial H}{\partial \Psi^{(1)}}\,d\Psi^{(1)}\right)$$

- Obviously, the **Requirement** can be satisfied only by

$$g^{\Phi\Phi} = \kappa \, \gamma \, L$$

- The autonomous necessary condition for $H(\psi, \Psi^{(1)})$ is:

$$H \left( \frac{1}{4\psi^2} H - \frac{1}{2\psi} \frac{\partial H}{\partial \psi} - \kappa \right) = 0,$$

which upon integration gives

$$
\begin{aligned}
H &= 0, \\
H &= -\frac{4\kappa\psi^2}{3} + \sqrt{\psi} \, a(\Psi^{(1)})
\end{aligned}
$$

where $a(\Psi^{(1)})$ is an arbitrary function of its argument.

- We, finally, end up with the two scalar functionals:

$$y^1 = \int \gamma(\tilde{r})d\tilde{r}, \qquad y^2 = \int \gamma(\tilde{r})\psi(\tilde{r})^2 d\tilde{r}$$

inducing the following metric:

$$g_{ren}^{AB}(y^1, y^2) = -\frac{1}{4}\begin{pmatrix} -\dfrac{\left(y^1\right)^2}{y^2} & y^1 \\ \\ y^1 & 3\,y^2 \end{pmatrix}$$

- To complete the investigation, we have to start with $y^1$, $y^2$ and consider a third functional of the general form:

$$y^3 = \int d\tilde{r}\,\gamma(\tilde{r})\,L(\Psi^{(1)})$$

- Any other functional, say

$$y^4 = \int d\tilde{r}\, \gamma(\tilde{r})\, K\left[\psi(\tilde{r}), \Psi^{(1)}(\tilde{r})\right],$$

  can be considered as a function of $y^1, y^2, y^3$.

- Indeed, since the scalar functions appearing in the integrands of $y^2, y^3$ form a base in the space spanned by $\psi, \Psi^{(1)}$, we can express the generic $K$ in $y^4$ as

$$K\left[\sqrt{\frac{\gamma\psi^2}{\gamma}}, \Psi^{(1)}\right],$$

  which (through the **Assumption**) gives

$$y^4 = y^1\, K\left[\sqrt{\frac{y^2}{y^1}}, L^{-1}\left(\frac{y^3}{y^1}\right)\right].$$

- It is interesting that the induced metric is

$$g_{ren}^{AB}(y^1, y^2, y^3) = -\frac{1}{4} \begin{pmatrix} -\frac{(y^1)^2}{y^2} & y^1 & -\frac{y^1 y^3}{y^2} \\ y^1 & 3\,y^2 & y^3 \\ -\frac{y^1 y^3}{y^2} & y^3 & -\frac{(y^3)^2}{y^2} + \frac{4\,(y^1)^2 F(\frac{y^3}{y^1})^2}{3\,y^2 F'\left[F(\frac{y^3}{y^1})\right]^2} \end{pmatrix}$$

where

$$g_{ren}^{33} = \int \frac{\gamma}{4\psi^2} W(L(\Psi^{(1)})) dr$$

or, equivalently, via the **Assumption**

$$g_{ren}^{33} = \frac{(y^1)^2}{4y^2} W(\frac{y^3}{y^1})$$

- The quantity $W(L(\Psi^{(1)}))$ is parameterized as

$$L\left(\Psi^{(1)}\right)^2 - \frac{4\, F[L\left(\Psi^{(1)}\right)]^2}{3\, F'[F[L\left(\Psi^{(1)}\right)]]^2}\,.$$

  and $g_{ren}^{AB}$ satisfies the **Requirement** for any choice of $L(\Psi^{(1)})$.

- More interestingly and unexpectedly, the freedom in the choice of $L$ translates into a pure g.c.t. freedom of the above renormalized metric.

The quantum analogue of the quadratic constraint is taken as:

$$\hat{\mathcal{H}}_o \Psi \equiv [-\frac{1}{2} \Box_c + V_{ren}] \Psi(Y^1, Y^2, Y^3) = 0$$

with

$$\Box_c = \Box + \frac{d-2}{4(d-1)} R,$$

where $(Y^1, Y^2, Y^3)$ are given by the transformation

$$(y^1, y^2, y^3) =$$
$$(e^{-\frac{1}{8}(5Y^1+3Y^3)}, e^{Y^1+Y^2+Y^3}, e^{-\frac{1}{8}(5Y^1+3Y^3)} F^{-1}(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)}$$

and the renormalized metric is now given by:

$$g_{AB ren}(Y^1, Y^2, Y^3) = \begin{pmatrix} e^{Y^1+Y^2+Y^3} & 0 & 0 \\ 0 & -\frac{4}{3} e^{Y^1+Y^2+Y^3} & 0 \\ 0 & 0 & -e^{Y^1+Y^2+Y^3} \end{pmatrix}$$

The renormalized potential is

$$V_{ren} = -2\,\epsilon\,e^{-\frac{1}{8}(5\,Y^1+3\,Y^3)} -$$
$$-2\,e^{-\frac{1}{8}(5\,Y^1+3\,Y^3)}\left[L^{-1}\left(F^{-1}(e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)})\right)\right]^2 +$$
$$+2\,\Lambda\,e^{Y^1+Y^2+Y^3} + A_T.$$

Exploiting the previously mentioned freedom in the choice of $L$, we can simplify the potential by the choice

$$L(\Psi^{(1)}) = m + \int \frac{(\Psi^{(1)})^{3/2}}{((\Psi^{(1)})^2 - \epsilon)^{13/16}}\,e^{k-\frac{3\,\epsilon}{16((\Psi^{(1)})^2-\epsilon)}}\,d\Psi^{(1)}$$

(where $c_1\,m + c_2 + c_3\,e^k = 0$)
which satisfies the relation:

$$F^{-1}\left(e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)}\right) = L\left(\sqrt{e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)} - \epsilon}\right)$$

Gathering all the pieces together, we end up with the following Wheeler-deWitt equation:

$$2\,\Lambda\,e^{2(Y^1+Y^2+Y^3)}\Psi(Y^1,Y^2,Y^3) - 2\,e^{\frac{4}{3}Y^2}\Psi(Y^1,Y^2,Y^3) +$$

$$A_T\,e^{Y^1+Y^2+Y^3}\Psi(Y^1,Y^2,Y^3) - \frac{3}{128}\Psi(Y^1,Y^2,Y^3) -$$

$$\frac{1}{4}\frac{\partial\Psi(Y^1,Y^2,Y^3)}{\partial Y^1} + \frac{3}{16}\frac{\partial\Psi(Y^1,Y^2,Y^3)}{\partial Y^2} +$$

$$\frac{1}{4}\frac{\partial\Psi(Y^1,Y^2,Y^3)}{\partial Y^3} - \frac{1}{2}\frac{\partial^2\Psi(Y^1,Y^2,Y^3)}{\partial\left(Y^1\right)^2} +$$

$$\frac{3}{8}\frac{\partial^2\Psi(Y^1,Y^2,Y^3)}{\partial\left(Y^2\right)^2} + \frac{1}{2}\frac{\partial^2\Psi(Y^1,Y^2,Y^3)}{\partial\left(Y^3\right)^2} = 0.$$

which is readily solved via separation of variables

$$\Psi(Y^1, Y^2, Y^3) = \Psi^1(Y^1)\,\Psi^2(Y^2)\,\Psi^3(Y^3)$$

leading to the solutions:

$$
\begin{aligned}
\Psi^1(Y^1) &= c_1\, e^{\frac{1}{4}\left(-1-\sqrt{1+32\,m}\right)Y^1} + c_2\, e^{\frac{1}{4}\left(-1+\sqrt{1+32\,m}\right)Y^1}\\
\Psi^2(Y^2) &= c_3\, e^{-Y^2/4}\, I_{-\frac{\sqrt{3}}{8}\sqrt{3+128\,n}}\left(2\sqrt{3}\,e^{2\,Y^2/3}\right)\\
&\quad + c_4\, e^{-Y^2/4}\, I_{\frac{\sqrt{3}}{8}\sqrt{3+128\,n}}\left(2\sqrt{3}\,e^{2\,Y^2/3}\right)\\
\Psi^3(Y^3) &= c_5\, e^{\frac{1}{8}\left(-2-\sqrt{7+128\,m-128\,n}\right)Y^3} + c_6\, e^{\frac{1}{8}\left(-2+\sqrt{7+128\,m-128\,n}\right)Y^3}
\end{aligned}
$$

# Summary

- The proper imposition of quantum analogues of the linear (momentum) constraint reduces an initial collection of state vectors to all scalar smooth functionals.

- The demand that the midi-superspace metric must define on the space of these states an induced metric whose components are given in terms of the same states, which is made possible through an appropriate renormalization **Assumption** and **Requirement**, severely reduces possible state vectors to three unique smooth scalar functionals.

- The quantum analogue of the Hamiltonian constraint produces a Wheeler-deWitt equation based on the reduced manifold of states, which is completely integrated.

# Open Questions

- Before a probability can be assigned to each of these geometries, two problems remain to be solved:
  - (1) Render finite the three smooth functionals $y^1, y^2, y^3$.
  - (2) Select an appropriate inner product.
- The first will need a final regularization of $y^1, y^2, y^3$, but the detailed way to do this will depend upon the particular geometry under consideration.
- For the second, a natural choice could be the determinant of the induced renormalized metric, although the problem with the positive definiteness may dictate another choice.

# Conceptual Query

- If the above quantization scheme trully achieves general coordinate invariant characterization of the wave function, then there must be a way to classify classical geometries using only **first derivatives** of the metrics.

- The current state of knowledge for the subject is the Cartan-Karlhede equivalence classification scheme, which requires up to seven derivatives of the Riemann tensor.

- BUT... we are working on it!!!!!!!!!!!!!