I. BAKAS

INSTANTONS OF HOŘAVA-LIFSHITZ GRAVITY

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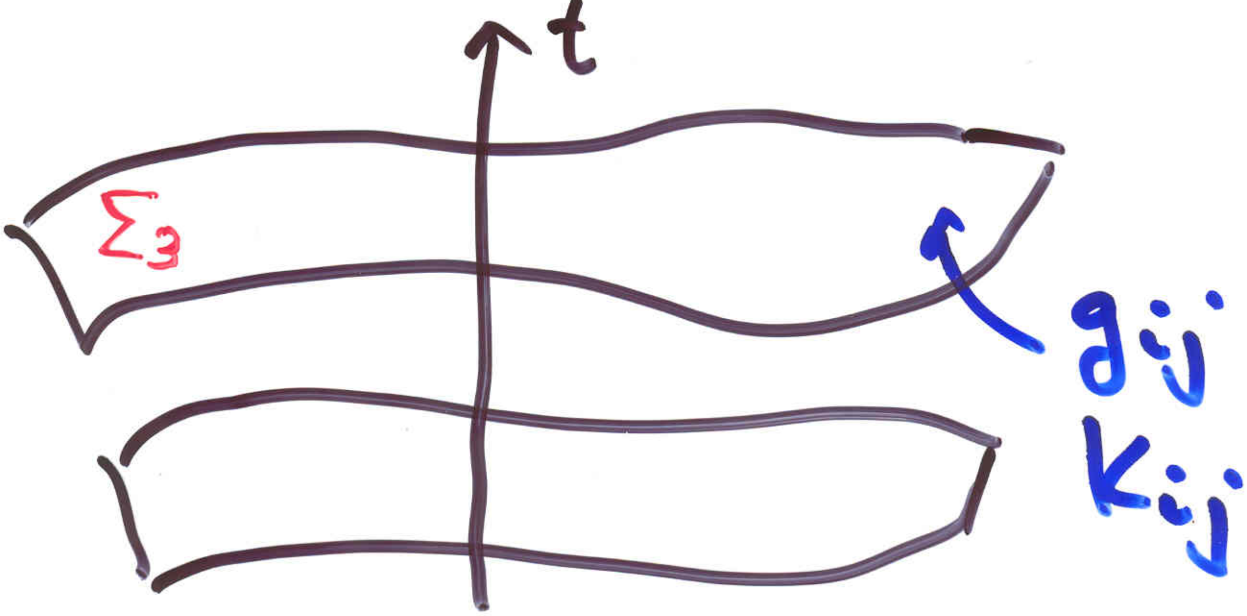
HOŘAVA-LIFSHITZ GRAVITY

Consider ADM decomposition

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$



g_{ij} : Riemannian metric on Σ_3

N : lapse function

N^i : shift functions

K_{ij} : extrinsic curvature

$$K_{ij} = \frac{1}{2N}\left(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

Also, consider superspace of all g_{ij} on Σ_3 endowed with metric

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}$$

and its inverse

$$G_{ijkl} = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk}\right) - \frac{\lambda}{3\lambda-1} g_{ij}g_{kl}$$

so that

$$G^{ijkl}G_{klmn} = \frac{1}{2}\left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m\right).$$

- λ can take any real value
- G^{ijkl} is +ive definite for $\lambda < \frac{1}{3}$
- in GR we have $\lambda = 1$ (DeWitt)

The action is $S = S_K - S_V$

where

$$S_K = \frac{2}{\kappa^2} \int dt\, d^3x \sqrt{g}\, N K_{ij} G^{ijkl} K_{kl}$$

$$= \frac{2}{\kappa^2} \int dt\, d^3x \sqrt{g}\, N \left(K_{ij} K^{ij} - \lambda K^2 \right)$$

and

$$S_V = \frac{\kappa^2}{2} \int dt\, d^3x \sqrt{g}\, N\, V[g]$$

- For Einstein gravity

$$V = -R + 2\Lambda$$

- For Hořava-Lifshitz gravity

$$V = E^{ij} G_{ijkl} E^{kl}$$

$$E^{ij} = -\frac{1}{2\sqrt{g}} \frac{\delta W}{\delta g_{ij}}$$

i.e., W is superpotential functional (detailed balance)

- W is taken to be the action of 3-d (massive) gravity on Σ_3
- Compare with point-particle dynamics derived from action

$$S = \int dt \left[\frac{1}{2}\dot{q}^2 - \left(\frac{\partial W}{\partial q}\right)^2 \right]$$

but now this is defined in ∞-dim space of all metrics on Σ_3.

A particular choice of $W[g]$ is topologically massive gravity

$$W = \frac{2}{\kappa_w^2}\int d^3x\sqrt{g}\,(R - 2\Lambda_w) + \frac{1}{\omega} W_{CS}$$

where W_{CS} is grav. CS term

$$W_{CS} = \frac{1}{2}\int_{\Sigma_3} \mathrm{Tr}\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right)$$

written in terms of connection 1-forms.

Then,

$$E^{kl} = \frac{1}{\kappa_w^2}\left(R^{kl} - \frac{1}{2}Rg^{kl} + \Lambda_w g^{kl}\right)$$

$$- \frac{1}{\omega}\frac{\varepsilon^{ijk}}{\sqrt{g}}\nabla_i\left(R_j^l - \frac{1}{4}R\delta_j^l\right)$$

is sum of Einstein and Cotton tensors.

In turn, HL potential takes the form

$$V = \frac{\kappa^2}{2w} C^{ij} C_{ij} - \frac{\kappa^2}{w\kappa_w^2} C^{ij} R_{ij} + \frac{\kappa^2}{2\kappa_w^4}\left(R_{ij}R^{ij} - \frac{4\lambda-1}{4(3\lambda-1)} R^2\right) + \frac{\kappa^2 \Lambda_w}{2(3\lambda-1)\kappa_w^4}\left(R - 3\Lambda_w\right)$$

Cotton tensor C_{ij} is 3rd order and so V has up to 6-derivatives

$\Rightarrow$ anisotropic scaling parameter $z = 3$

HL gravity is non-relativistic and was proposed to serve as UV completion of 4-d GR by specific higher curvature terms (power-counting renormalizable theory).

Projectable case

$$N = N(t) , \quad N^i = N^i(t, x)$$

leads to invariance under foliation preserving diffeos. We'll choose later

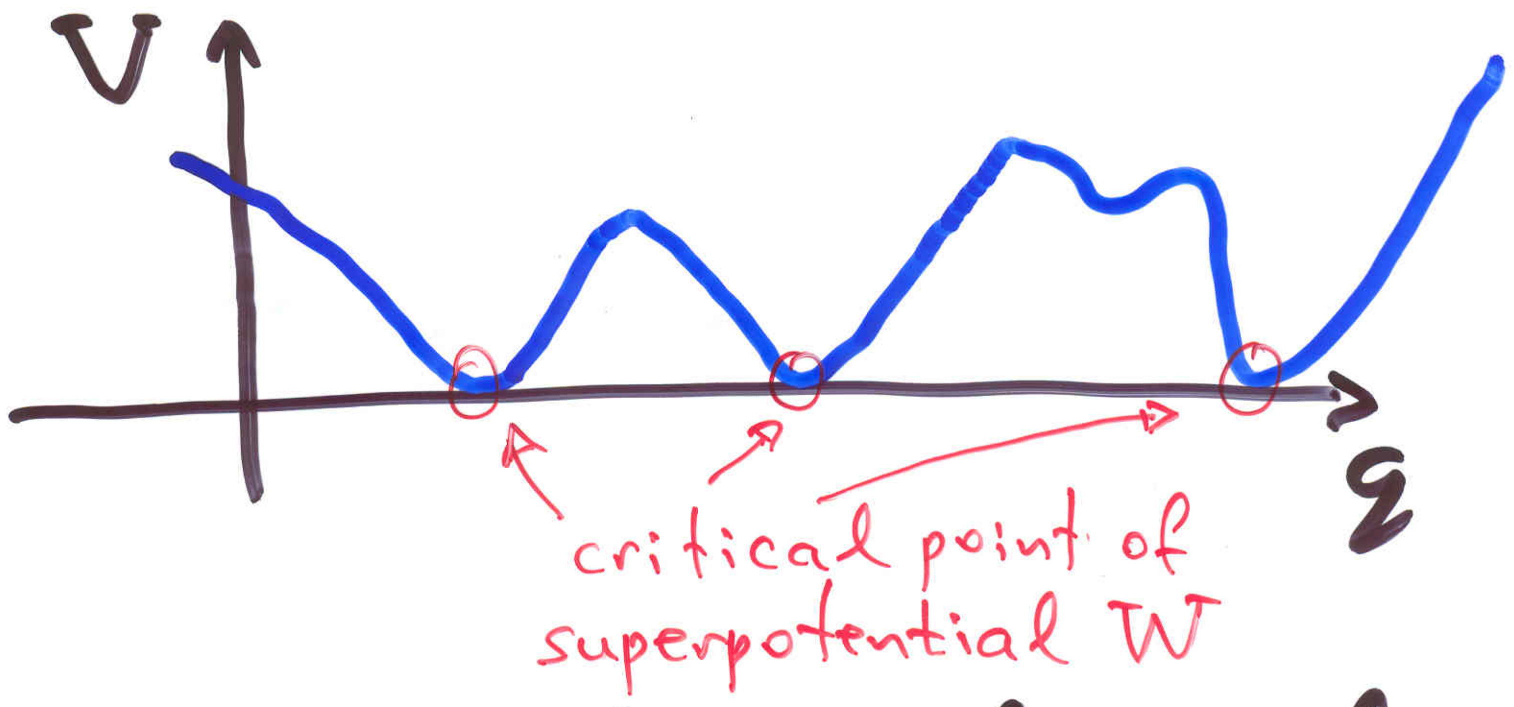
$$N = 1 , \quad N^i = 0 .$$

INSTANTON SOLUTIONS

Consider point-particle system

$$S = \int dt \left[\frac{1}{2} \dot{q}^2 - \left(\frac{\partial W}{\partial q} \right)^2 \right]$$

with potential



Critical points are classical solutions, but there are also instanton solutions interpolating between critical points

Let $t \to it$ and find lower bound of Euclidean action

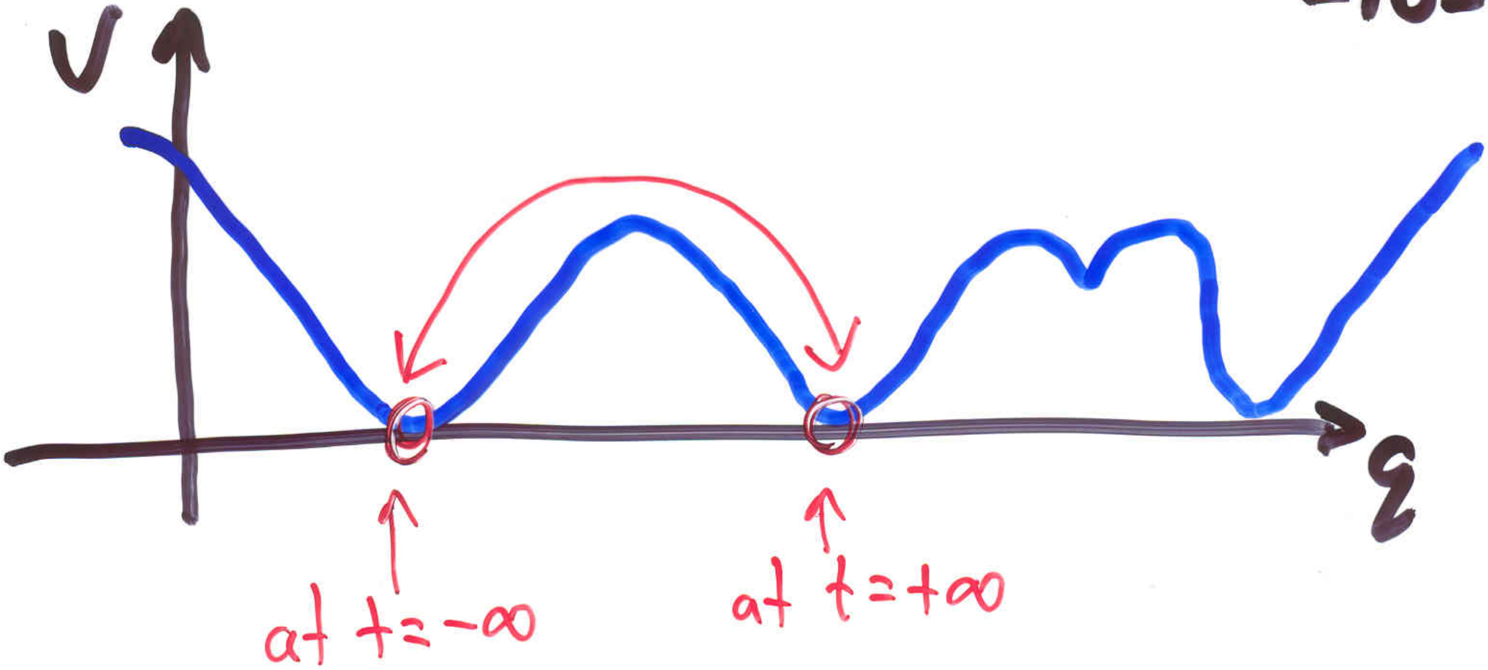
$$S_E = \int_{-\infty}^{+\infty} dt \left[\frac{1}{2}\dot{q}^2 + \left(\frac{\partial W}{\partial q}\right)^2 \right] =$$

$$= \int_{-\infty}^{+\infty} dt \left(\frac{1}{\sqrt{2}}\dot{q} \pm \frac{\partial W}{\partial q} \right)^2 \mp$$

$$\mp \sqrt{2} \int_{-\infty}^{+\infty} dt \, \dot{q}\frac{\partial W}{\partial q} = \frac{dW}{dt}$$

$$\geq \mp \sqrt{2} \left(W(t=+\infty) - W(t=-\infty) \right)$$

Saturation of lower bound provides instanton solutions with finite Euclidean action:



Instanton equations are 1st order in time,

$$\dot{q} = \mp \sqrt{2}\, \frac{\partial W}{\partial q}$$

and their action is

$$S^{inst} = \sqrt{2}\, |\Delta W| > 0$$

Have instantons (+ sign) and anti-instantons (− sign) related by $t \to -t$.

Applying this to Euclidean version of HL gravity we have (setting $N=1$, $N^i=0$):

$$S_E = \frac{2}{\kappa^2}\int dt\, d^3x \sqrt{g}\left(K_{ij} \pm \frac{\kappa^2}{2} G_{ijmn} E^{mn}\right) \times G^{ijkl}\left(K_{kl} \pm \frac{\kappa^2}{2} G_{klrs} E^{rs}\right)$$

$$\mp 2\int dt\, d^3x \sqrt{g}\, K_{ij} E^{ij}$$

Thus,

$$S_E \geq \mp 2 \int dt\, d^3x \sqrt{g}\, K_{ij} E^{ij} =$$

$$= \mp \int dt\, d^3x \sqrt{g}\, E^{ij} \partial_t g_{ij}$$

$$= \pm \frac{1}{2}\int dt \frac{dW}{dt} = \frac{1}{2}|\Delta W|$$

Implicit assumptions:

- $\lambda < \frac{1}{3}$ so that superspace metric is +ive definite
- $\partial \Sigma_3 = \emptyset$, e.g. $\Sigma_3 \simeq S^3$, to avoid other boundary terms

Then, the HL equations are:

$$K_{ij} = \mp \frac{\kappa^2}{2} G_{ijmn} E^{mn}$$

which in turn yield

$$\partial_t g_{ij} = \pm \frac{\kappa^2}{2\sqrt{g}} G_{ijkl} \frac{\delta W[g]}{\delta g_{kl}}$$

which are gradient flow equations

Choosing, in particular, $\tilde{W}$ of topologically massive 3d grav. we obtain

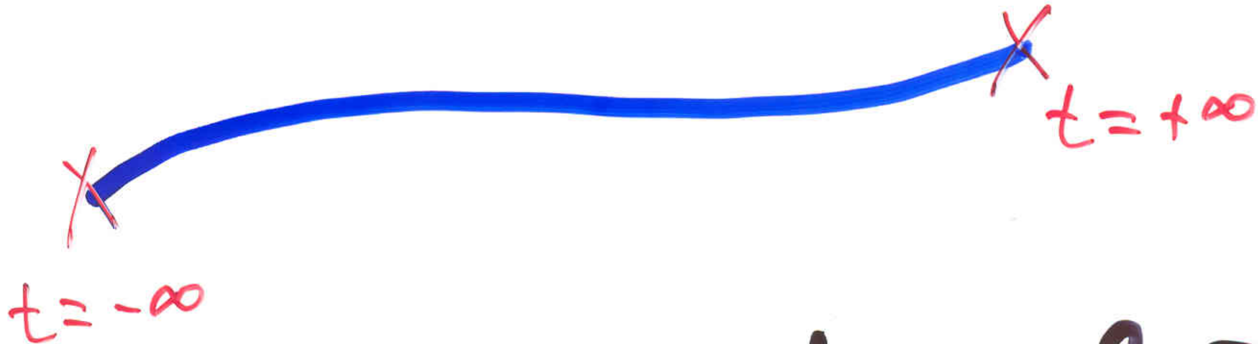
$$\partial_t g_{ij} = \frac{\kappa^2}{w} C_{ij} -$$

$$- \frac{\kappa^2}{\kappa_w^2}\left(R_{ij} - \frac{2\lambda-1}{2(3\lambda-1)} R g_{ij} + \frac{\Lambda_w}{1-3\lambda} g_{ij}\right)$$

termed as Ricci-Cotton flow.

- fixed points are vacua of TMG (independent of λ)
- in all cases, $\tilde{W}$ is an entropy functional of the flow as it changes monotonically

Need more than one fixed point to support instantons



and extrinsic curvature of Σ_3 vanishes at end points.

HL instantons are eternal solutions of geometric flows on Σ_3.

Singularities of flow lines lead to incomplete space-time metrics.

As application, consider $\widetilde{SU}(2)$ instantons using Bianchi IX ansatz for g_{ij} on S^3. Obtain consistent reduction to system of ODE and find:

- $z=2$ HL theory admits no SU(2) instantons (as Ricci flow has only one fixed point)
- $z=3$ HL theory admits SU(2) instantons that are all classified in BBLP.
- one can also consider $z>3$.

Onsager-Machlup theory

Introduced in the 50's to model non-equilibrium thermodynamics

OM functional is HL but more general:

$$\Phi^I \rightsquigarrow g_{ij}$$

$$\mathcal{O}^{IJ} \rightsquigarrow G^{ijkl}$$

$$W \rightsquigarrow W$$

Thermodynamic (entropic) force is $\frac{\delta W}{\delta \Phi^I}$ driving the flow.

Alternative view of HL theory as toy model to study landscape of a given theory W (in general spirit of OM).

- instantons provide transitions among vacua
- W is entropy and can be used to assign probabilities
- try to embed in string or M-theory and view flows as RG eqs