Horizons, Singularities and Causal Structure of the Generalized McVittie Spacetimes

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1. Line Elements

- Schwarzschild metric in isotropic coordinates:

$$ds^2 = -(1+\frac{m_0}{2r})^{-2}(1-\frac{m_0}{2r})^2 dt^2 + (1+\frac{m_0}{2r})^4(dr^2 + r^2 d\Omega^2)$$

[$d\Omega^2$ is the metric on the unit 2-sphere]

- McVittie spacetime (1933):

$$ds^2 = -(1+\frac{m_0}{2ra(t)})^{-2}(1-\frac{m_0}{2ra(t)})^2 dt^2 + (1+\frac{m_0}{2ra(t)})^4 a^2(t)(dr^2 + r^2 d\Omega^2)$$

A Schwarzschild black hole embedded in an FLRW spacetime?

- Generalized McVittie (GMcV) spacetimes (Faraoni and Jacques, 2007):

$$\boxed{ds^2 = -(1+\frac{m(t)}{2r})^{-2}(1-\frac{m(t)}{2r})^2 dt^2 + (1+\frac{m(t)}{2r})^4 a^2(t)(dr^2 + r^2 d\Omega^2)}$$

2. Special Cases of the GMcV Spacetimes

• The original McVittie spacetime is recovered if $m(t) = m_0/a(t)$.

[This is equivalent to a "no-accretion" condition: see Section 3.]

• If $m(t) = m_0/a(t)$ and $a(t) = \exp\left(\sqrt{\Lambda/3}\, t\right)$, a transformation of the form

$$\bar{r} = ar\left(1 + \frac{m_0}{2ra(t)}\right)^2, \quad d\bar{t} = dt + \sqrt{\Lambda/3}\,\bar{r}\left(1 - \frac{2m_0}{\bar{r}}\right)^{-1/2}\left(1 - \frac{2m_0}{\bar{r}} - \frac{1}{3}\Lambda\bar{r}^2\right)^{-1} d\bar{r}$$

reduces the McVittie line element to:

$$ds^2 = -\left(1 - \frac{2m_0}{\bar{r}} - \frac{1}{3}\Lambda\bar{r}^2\right)d\bar{t}^2 + \left(1 - \frac{2m_0}{\bar{r}} - \frac{1}{3}\Lambda\bar{r}^2\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

which is the Kottler (1918) or Schwarzschild-de Sitter / anti-de Sitter metric.

• If $m(t) = m_0$ and $a(t) = a_0 t^{2/3}$ the metric is of a form that Faraoni (2009) claimed to be locally isometric to the Sultana-Dyer (2005) solution. In fact, the Sultana-Dyer solution is not a GMcV spacetime (Sun, 2010).

3. Accretion in the GMcV Spacetimes

- Einstein tensor components for the GMcV spacetimes:

$$G_t^t = 3a^{-2}(2r-m)^{-2}[(2r+m)\dot{a}+2a\dot{m}]^2$$

$$G_r^t = 8a^{-1}(2r-m)^{-3}(2r+m)(\dot{a}m+a\dot{m})$$

$$G_r^r = G_\theta^\theta = G_\phi^\phi = 3a^{-2}(2r-m)^{-2}[(2r+m)\dot{a}+2a\dot{m}]^2$$
$$+2(2r-m)^{-1}(2r+m)\frac{\partial}{\partial t}[a^{-1}(2r-m)^{-1}\{(2r+m)\dot{a}+2a\dot{m}\}]$$

- McVittie (1933) generated his solutions by imposing the "no-accretion" condition $G_r^t = 0$, which is equivalent to the constraint $am = m_0$.

- The Misner-Sharp (1964) mass *M* defined by

$$1-2M/\bar{r} = \nabla^a\bar{r}\,\nabla_a\bar{r} \qquad \text{where} \qquad \bar{r} = (1+\tfrac{m}{2r})^2 ar$$

has the value

$$M = ma + \frac{1}{128}r^{-3}a(2r-m)^{-2}(2r+m)^6[(2r+m)\dot{a}+2a\dot{m}]^2$$

$\Rightarrow M(r=m/2)$ is uniquely finite in the McVittie solution, but is not constant.

4. Stress-Energy Source (Perfect Fluid)

• In a general spherically symmetric spacetime:

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)(dr^2 + r^2 d\Omega^2)$$

the stress-energy tensor of a perfect fluid is:

$$T^a_b = (\rho + p)u^a u_b + p\delta^a_b$$

where ρ is density, *p* is pressure, and the bulk 4-velocity is

$$u^a = (A^{-1}\cosh\beta, B^{-1}\sinh\beta, 0,0)^a \qquad u_b = (-A\cosh\beta, B\sinh\beta, 0,0)_b$$

• The Einstein equation $G^a_b = -8\pi T^a_b$ then implies for the GMcV spacetimes that

$$\rho\sinh^2\beta + p\cosh^2\beta = T^r_r = T^\theta_\theta = p$$

and $$A^{-1}B(\rho + p)\sinh\beta\cosh\beta = T^t_r \propto \frac{d}{dt}(ma)$$

The first equation is satisfied only if $(\rho + p)\sinh\beta = 0$. So a GMcV spacetime with a perfect fluid source must have $T^t_r = 0$, $ma = m_0$ and be in the McVittie class.

5. Stress-Energy Source (Heat Flux)

- Faraoni and Jacques (2007) proposed an imperfect fluid source:

$$T^a_b = (\rho + p)u^a u_b + p\delta^a_b + q^a u_b + u^a q_b$$

where $q^a = (0, B^{-1}q, 0, 0)^a$ is a heat flux vector.

- Then $$\rho\sinh^2\beta + p\cosh^2\beta + 2q\sinh\beta = T^r_r = T^\theta_\theta = p$$

and $$A^{-1}B[(\rho + p)\sinh\beta + q]\cosh\beta = T^t_r = -\frac{1}{8\pi}G^t_r$$

The first equation implies that

$$q = -\tfrac{1}{2}(\rho + p)\sinh\beta$$

and the second that

$$\frac{d}{dt}(ma) = -\frac{\pi}{8r^2}(4r^2 - m^2)^2(\rho + p)\sinh\beta\cosh\beta$$

• Null energy condition:

$$T_b^a v_a v^b \geq 0 \qquad \text{with} \qquad v^a = (A^{-1}, B^{-1}, 0,0)^a \qquad v_b = (-A, B, 0,0)_b$$

$$\Rightarrow \qquad T_r^r - T_t^t = (\rho + p)\cosh^2\beta \geq 0$$

or equivalently $\frac{d}{dt}(\dot{a}/a) \leq 0$ and $\dot{m}\frac{d}{dt}(ma) \leq 0$ and

either $a\frac{d^2}{dt^2}(ma) \leq \dot{a}\frac{d}{dt}(ma)$

or $[a\frac{d^2}{dt^2}(ma) - \dot{a}\frac{d}{dt}(ma)]^2 \leq 2a^3\dot{m}\frac{d}{dt}(ma)\frac{d}{dt}(\dot{a}/a)$

• So $q > 0$ and ma is increasing if the fluid velocity is inwards ($\sinh\beta < 0$). Also, if the Universe is expanding ($\dot{a} \geq 0$) then $\dot{m} \leq 0$ and $\frac{d}{dt}(ma) \geq 0$.

• The condition $1 > |\tanh\beta| = |2A^{-1}BG_r^t/(G_r^r - G_t^t)|$ imposes even more complicated conditions on m and a.

6. Singularities

- Assuming that m and a are positive and C^∞ for all $t\in(0,\infty)$, scalar divergences can only occur at

$$t=0,\infty \qquad r=0,\pm\infty \qquad \text{and} \qquad r=\pm m(t)/2.$$

- The Ricci scalar is:

$$\boxed{\begin{aligned}R=&-6a^{-2}(2r-m)^{-3}[(2r-m)\{(2r+m)^2(a\ddot{a}+\dot{a}^2)+2(2r+m)a^2\ddot{m}\\&+6a^2\dot{m}^2+4(3r+2m)a\dot{a}\dot{m}\}+8ra\dot{m}(2r\dot{a}+a\dot{m})]\end{aligned}}$$

It diverges at $r=m/2$ unless $ma\dot{m}(m\dot{a}+a\dot{m})=0$.

- If $(m\dot{a}+a\dot{m})=0$ the McVittie solution is recovered ($m=m_0/a$) and

$$\boxed{R=-6a^{-2}(2ar-m_0)^{-1}[(2ar+m_0)a\ddot{a}+(2ar-3m_0)\dot{a}^2]}$$

R now diverges at $r=m/2$ unless $a\ddot{a}-\dot{a}^2=0 \;\Rightarrow\; a(t)\propto e^{kt}$ (Kottler solution).

- If $\dot{m}=0$ then $\boxed{R=-6a^{-2}(2r-m)^{-2}(2r+m)^2(a\ddot{a}+\dot{a}^2)}$

R now diverges at $r=m/2$ unless $a\ddot{a}+\dot{a}^2=0 \;\Rightarrow\; a(t)\propto(t-t_0)^{1/2}$

If $a(t)=t^{1/2}$ then $R\equiv 0$ but $R^{ab}R_{ab}$ diverges at $r=m/2$, as

$$R^{ab}R_{ab}=t^{-4}(2r-m)^{-4}[\tfrac{3}{4}(2r+m)^4-512(2r+m)^{-4}m^2r^4t]$$

7. The Singular Surface at $r = m/2$

- In the limit as $r \to m/2$ the GMcV line element reduces to the metric of a spacelike (or null, if $\dot{m} = 0$) hypersurface:

$$ds^2 = 4a(t)^2[\dot{m}(t)^2 dt^2 + m(t)^2 d\Omega^2]$$

- The density ρ diverges as $\boxed{\frac{3}{2\pi}a^{-2}(2r-m)^{-2}[\frac{d}{dt}(ma)]^2}$ unless $\frac{d}{dt}(ma) = 0$.

- The pressure p diverges as $\boxed{-\frac{1}{\pi}a^{-1}(2r-m)^{-3}m\dot{m}\frac{d}{dt}(ma)}$ unless $\dot{m}\frac{d}{dt}(ma) = 0$.

[As $(2r-m)^{-1}$ if $\frac{d}{dt}(ma) = 0$ unless $\frac{d}{dt}(\dot{a}/a) = 0$; as $(2r-m)^{-2}$ if $\dot{m} = 0$ unless $a \propto t^{2/3}$.]

- The heat flux q diverges as $\boxed{\frac{1}{4\pi}a^{-1}(2r-m)^{-2}\frac{d}{dt}(ma)}$ unless $\frac{d}{dt}(ma) = 0$.

- The fluid speed tanhβ goes to zero as $\boxed{\frac{1}{2am\dot{m}}(2r-m)}$ unless $\dot{m} = 0$.

- In the McVittie case, Kaloper, Kleban and Martin (2010) describe the surface $r = m/2$ as an “inhomogeneous Big Bang” singularity.

8. Apparent Horizons

- Future-directed radial null geodesics:

$$u_\pm^a = \kappa_\pm v_\pm^a \qquad \text{where} \qquad v_\pm^a = (A^{-1}, \pm B^{-1}, 0, 0)^a$$

and the conformal factors $\kappa_\pm > 0$ are determined by the affine conditions

$$u_\pm^b D_b u_\pm^a = 0$$

- The radial null geodesic expansions $\theta_\pm = D_a u_\pm^a$ are given by

$$\theta_\pm/\kappa_\pm = 2[B_t \pm (AB^{-1}B_r + r^{-1}A)]$$

$\Rightarrow$

$$\boxed{\begin{gathered}\theta_\pm \propto m^3(2\phi - \dot{a}m) + 4m^2(3\phi - \dot{a}m \pm 1)r + 8m(3\phi \mp 2)r^2 \\ +16(\phi + \dot{a}m \pm 1)r^3 + 16\dot{a}r^4\end{gathered}}$$

where $\phi = \frac{d}{dt}(ma)$.

- In terms of $\alpha = \dot{a}m$ and $x = (2r - m)/m$,

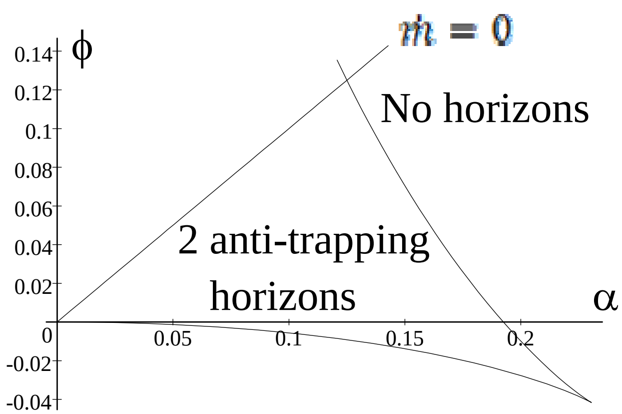
$$\boxed{\theta_\pm \propto 16\phi + 8(3\phi + \alpha)x + 2(6\phi + 6\alpha \pm 1)x^2 + 2(\phi + 3\alpha \pm 1)x^3 + \alpha x^4}$$

• So if $\dot{\alpha} > 0$ and (n.e.c.) $\phi > 0$ then:

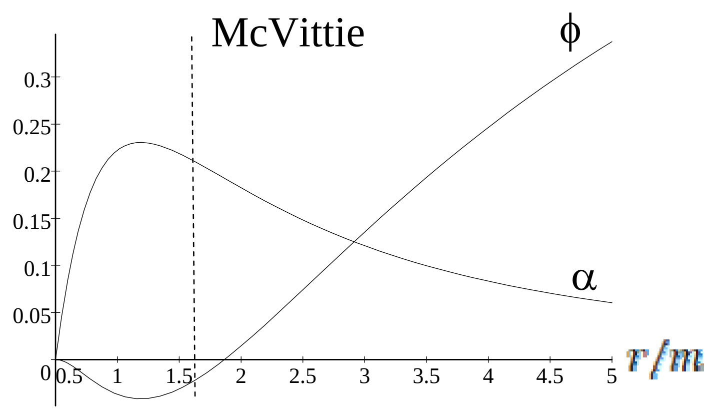
- ♦ $\theta_{\pm} > 0$ at $r = (m/2)^{+}$ and for $r \to \infty$
- ♦ θ_{+} has no zeroes with $r > m/2$
- ♦ θ_{-} has either 0 or 2 zeroes with $r > m/2$

$\Rightarrow$ (At a given *t*) either there is no apparent horizon, or there are 2 anti-trapping horizons – where $\theta_{+} > 0$ and θ_{-} changes from < 0 to > 0.

Horizons in parameter space



Degenerate horizon radius



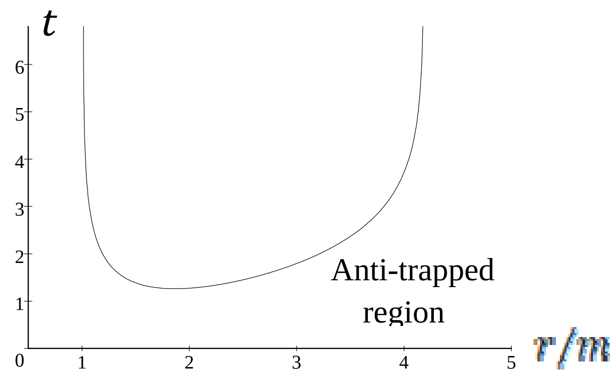
[Null energy condition: $\phi > 0$ and $\dot{m} < 0$ ($\Rightarrow$ $\phi < \alpha$)]

[McVittie solutions have $\phi = 0$.]

A McVittie solution

$$\alpha = \tfrac{3}{20}(1 + e^{-t}) \quad \text{and} \quad \phi = 0$$

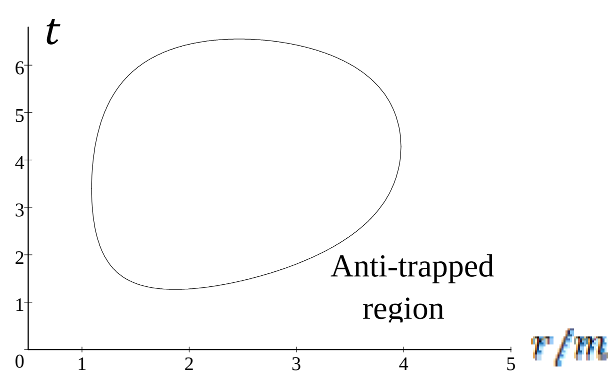
$$\Rightarrow \quad a(t) = \exp\{\tfrac{3}{20}(t - e^{-t} + 1)\},\ m(t) = 1/a(t)$$



A GMcV solution

$$\alpha = \tfrac{3}{20}(1 + e^{-t}) \quad \text{and} \quad \phi = 10^{-4} e^{t}$$

$$\Rightarrow \quad a(t) = \exp\{\tfrac{3\times 10^{4}}{40}(3 - 2e^{-t} - e^{-2t})\},\ m(t) = 10^{-4} e^{t}/a(t)$$



9. Null Geodesic Completeness

- Affine lapse along radial null geodesics:

$$\Delta\lambda = \int (A/\kappa_\pm)\, dt$$

where

$$\frac{d}{dt}\kappa_\pm = -\kappa_\pm B^{-1}(B_t \pm A_r) \qquad \text{and} \qquad \frac{dr}{dt} = \pm A/B$$

$\Rightarrow$

$$\boxed{\frac{dx}{dt} = (\mathrm{am})^{-1}(2+x)^{-3}(1+x)[8\xi + 2(6\xi \pm 1)x + 2(3\xi \pm 1)x^2 + \xi x^3]}$$

with $\xi = \alpha - \phi \geq 0$; and

$$\boxed{\begin{array}{c}\frac{d}{dt}(\ln\kappa_\pm) = -(\mathrm{am})^{-1}(2+x)^{-4}[16\phi \pm 4 + 8(3\phi + \alpha \pm 1)x \\ +4(3\phi + 3\alpha \pm 1)x^2 + \alpha x^3]\end{array}}$$

[Note that $\frac{dx}{dt} > 0$ and $\frac{d}{dt}(\ln\kappa_\pm) \leq 0$ along all future-directed <u>outgoing</u> radial null geodesics, and so $A/\kappa_\pm \equiv (2+x)^{-1}x/\kappa_\pm$ is bounded below, and $\Delta\lambda = \int (A/\kappa_\pm)\, dt$ diverges.]

Hence:

• $\Delta\lambda$ is finite along any null curve terminating at a singular point

$$(t,r)=(t_0,m(t_0)/2)$$

(unless *m* is constant: then $r=m/2$ is null as in Schwarzschild, and $t_0=\infty$)

• If $\lim_{t\to 0^+} m(t)>0$ and $\lim_{t\to 0^+} a(t)=0$,
$\Delta\lambda$ is finite along any null curve terminating at a singular point

$$(t,r)=(0,r_0)$$

provided that $\lim_{t\to 0^+} 1/\dot{a}(t)=0$ (otherwise $r_0=\infty$)

• $\Delta\lambda$ is finite along any null curve terminating at a point on the apparent horizon.

• <u>Outgoing null geodesics</u>: If $\lim_{t\to\infty} m(t)=0$ then $x\to\infty$ as m^{-1} (*r* is bounded).
If $\lim_{t\to\infty} m(t)=m_\infty>0$ then *x* and *r* are bounded if $\lim_{t\to\infty}\dot{a}(t)=\infty$; otherwise *x* and *r* both go to ∞ as $t\to\infty$.

In all cases $\Delta\lambda$ diverges and the geodesics are future-complete.

• <u>Ingoing null geodesics</u>: If $\xi_\infty \equiv \lim_{t\to\infty}(\alpha - \phi) > \sqrt{3}/9$ then the behavior is qualitatively the same as for the outgoing null geodesics.

$\boxed{\Delta\lambda \text{ diverges and the geodesics are future-complete.}}$

If $\xi_\infty < \sqrt{3}/9$ then $\frac{dx}{dt} \to 0$ as $x \to x_-$ and x_+, with $x_- < 1 + \sqrt{3} < x_+$.
Ingoing geodesics with $x > x_+$ have the same qualitative behavior as the outgoing geodesics: they are again future-complete.
Ingoing geodesics with $x < x_-$ or $x_- < x < x_+$ asymptote to $x = x_-$ as $t \to \infty$.
If $\phi_\infty \equiv \lim_{t\to\infty}\phi < -(2 + x_-)^{-4}(1 + x_-)(x_-^2 - 2x_- - 2)$ then κ_- is guaranteed to go to zero more rapidly than t^{-1}.

$\boxed{\Delta\lambda \text{ is finite and the geodesics are future-incomplete.}}$

• The incompleteness of the ingoing radial null geodesics for a certain parameter range generalizes the result of Kaloper, Kleban and Martin (2010) that the McVittie solutions (which have $\phi = 0$) are geodesically incomplete if $\lim_{t\to\infty}\alpha < \sqrt{3}/9$

10. References

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- V. Faraoni and A. Jacques (2007), *Phys. Rev.* **D76**, 063510, arXiv: 0707:1350v1 [gr-qc].

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- N. Kaloper, M. Kleban and D. Martin (2010), “McVittie’s Legacy: Black Holes in an Expanding Universe”, arXiv: 1003:4777v1 [hep-th].

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For Sections 5 and 6 see also:

- M. Carrera and D. Giulini (2010), *Phys. Rev.* **D81**, 043521